Due 4/9:
G1 (present): page 150: 1, 7, 8
G2 (present): page 150: 3, 6, 9, 12, 14 (me: 3, 14)
All (turn in): page 150: 17, 19, 29, 36 (me)
Due 4/11:
Present: page 167: 20
All (turn in): page 167: 1, 22

# Page 150

## Exercise 3

**Let H = {0, ±3, ±6, ±9...}. Rewrite the condition $a^{-1}b \in H$ given in property 6 of the lemma on page 139 in additive notation. Assume that the group is Abelian. Use this to decide whether or not the following cosets of H are the same.**

Property 6: aH = bH iff $a^{-1}b \in H$
Rewritten: a + H = b + H iff $a^{-1} + b \in H$

a. **11 + H and 17 + H**: $-11 + 17 = 6 \in H$, so yes.

b. **−1 + H and 5 + H**: 1 + 5 = 6 mem H, so yes.

c. **7 + H and 23 + H**: $-7 + 23 = 16 \notin H$, so no.

## Exercise 14

**Let $C^*$ be the group of nonzero complex numbers under multiplication and let $H = \{a + bi \in C^* : a^2 + b^2 = 1\}$. Give a geometric description of the cosets (3 + 4i)H and (c + di)H.**
Well,
$(3 + 4i)H = \{(3 + 4i)h : h \in H\}$
$(3 + 4i)H = \{(3 + 4i)(a + bi) : a + bi \in C^*, a^2 + b^2 = 1\}$
$(3 + 4i)H = \{3a + 4ai + 3bi - 4b: a + bi \in C^*, a^2 + b^2 = 1\}$
$(3 + 4i)H = \{3a + (4a + 3b)i - 4b: a + bi \in C^*, a^2 + b^2 = 1\}$
thus,
$(c + di)H = \{ca + (da + cb)i - db: a + bi \in C^*, a^2 + b^2 = 1\}$
$(c + di)H = \{(ca - db) + (da + cb)i: a + bi \in C^*, a^2 + b^2 = 1\}$
It looks like the subset H just indicates the elements that create a unit circle.
When we multiply by some real constant > 1, we just get a coset that represents a bigger circle.
When we multiply by some complex constant (e.g. 2i), we just get a coset that represents a flipped circle (where x, y becomes y, x), and if the complex constant has a scaling factor (e.g. 2), then the circle grows by that factor.
I think the cosets scale it by $||c + di||$.

## Exercise 17

**Let G be a group with |G| = pq: p, q are prime. Prove that every proper subgroup of G is cyclic.**
Let H be a proper subgroup of G.
Since G is finite, |H| divides |G|.
Case:

i) |H| = 1: Then H is cyclic by default.

ii) $|H| \neq 1$: Then by the fundamental theorem of arithmetic, |H| = t: t $\in$ {p, q}

Notice: |H| > 1.

Let h $\in$ H: h $\neq$ e.

Then $1 < |< h >| \leq |H|$.

Since H is finite, $|< h >|$ divides |H|.

Since |H| is prime, its factors are only 1 and |H|.

Since $|< h >| \neq 1$, this implies that $|< h >| = |H|$.

Hence, H must be cyclic.

## Exercise 19

**Compute $5^{15}$ mod 7 and $7^{13}$ mod 11.**

Fermat's Little Theorem: For every integer a and prime p, $a^p \equiv a$ mod p.
And, if a is not divisible by p, then $a^{p-1} \equiv 1$ mod p (which is 1)
Also, if a and n are relatively prime, then $a^{\phi(n)}$ mod n $\equiv 1$ (from problem 18 in the book)

$$\begin{aligned} 5^{15} \mod 7 &= (5^7)(5^7)5^1 \mod 7 \\ &\equiv (5)(5)5 \mod 7 \text{ (by Fermat's Little Theorem)} \\ &\equiv 125 \mod 7 \\ &= 6 \end{aligned}$$

$$\begin{aligned} 7^{13} \mod 11 &= (7)^3(7)^{10} \mod 11 \\ &\equiv 7^3(1) \mod 11 \text{ (by Fermat's Little Theorem)} \\ &\equiv 147 \mod 11 \\ &= 4 \end{aligned}$$

## Exercise 29

**Let |G| = 33. What are the possible orders for the elements of G? Show that G must have an element of order 3.**

Each element of G must have an order of: 1, 3, 11, or 33 (since it generates a cyclic subgroup)
Let g $\in$ G. If |g| = 1, then g is the identity, which exists in every group.
|g| cannot be 33, since that's the size of the group. The maximum order for an element is n $-$ 1 where n is the size of the group. So the possible orders are 1, 3, and 11.
Let's suppose this group contains elements only of orders 1 and 11.
If we pick $g \in G : g \neq e$, then |g| = 11, and we have 11 elements accounted for so far.
Next, we pick $h \in G : h \notin \langle g \rangle$.
Thus, $\langle h \rangle$ generates another cyclic subgroup of order 11.
So far, we have accounted for 21 elements, since $\langle g \rangle \cap \langle h \rangle = e$.
However, we have 12 elements left, which we cannot cover with elements of order 11.
So, we must have an element of order 3.

### Exercise 36

**Let G be a group and |G| = 21. If g ∈ G and $g^{14}$ = e, what are the possibilities for |g|?**

Well, since g is a generator for H, a cyclic subgroup of G, that means that |H| must be a factor of |G|. Since |G| = 21 and 14 doesn't divide 21, |H| must be some factor of both 21 and 14, but lower than 14. Those possibilities are: 1, 7

## Page 167

### Exercise 1

**Prove that the external direct product of any finite number of groups is a group.**
Let $G_1$, $G_2$, ... $G_n$ be a finite collection of groups.
Then $G_1 \bigoplus G_2 \bigoplus ... \bigoplus G_n = \{(g_1, g_2, ... g_n) : g_i \in G_i\}$
and $(g_1, g_2, ... , g_n)(g_1{}', g_n{}', ... , g_n{}') = (g_1g_1{}', g_2g_2{}', ... , g_ng_n{}')$
Denote $D = \{(g_1, g_2, ... g_n) : g_i \in G_i\}$
Want to show that D is a group on the group product operation.
**Closure:**
Let a, b ∈ D.
So:
$a = (g_1, g_2, ... , g_n)$
$b = (g_1{}', g_2{}', ... , g_n{}')$
and
$ab = (g_1g_1{}', g_2g_2{}', ... , g_ng_n{}')$
Since $g_ig_i{}' \in G_i$ for i = 1, 2, .. n by definition of a group,
$(g_1g_1{}', g_2g_2{}', ... , g_ng_n{}') \in D$
**Associativity:**
Let a, b, c ∈ D.
So:
$a = (g_1, g_2, ... , g_n)$
$b = (g_1{}', g_2{}', ... , g_n{}')$
$c = (g_1{}'', g_2{}'', ... , g_n{}'')$
$(ab)c = (g_1g_1{}', g_2g_2{}', ... , g_ng_n{}')(g_1{}'', g_2{}'', ... , g_n{}'')$
$(ab)c = (g_1g_1{}' g_1{}'', g_2g_2{}' g_2{}'', ... , g_ng_n{}' g_n{}'')$
$(ab)c = (g_1, g_2, ... , g_n)(g_1{}' g_1{}'', g_2{}' g_2{}'', ... , g_n{}' g_n{}'') = a(bc)$
**Identity:**
Let $e = (e_1, e_2, ... , e_n)$ and let a ∈ D: $a = (g_1, g_2, ... , g_n)$
Notice:
$ae = (g_1, g_2, ... , g_n)(e_1, e_2, ... , e_n) = (g_1, g_2, ... , g_n)$
$ea = (e_1, e_2, ... , e_n)= (g_1, g_2, ... , g_n) = (g_1, g_2, ... , g_n)$
Hence, D contains an identity element: e
**Inverse:**
Let a ∈ D: $a = (g_1, g_2, ... , g_n)$
Define $a^{-1} = (g_1{}^{-1}, g_2{}^{-1}, ... , g_n{}^{-1})$
Notice:
$aa^{-1} = (g_1, g_2, ... , g_n)(g_1{}^{-1}, g_2{}^{-1}, ... , g_n{}^{-1}) = (g_1g_1{}^{-1}, g_2g_2{}^{-1}, ... , g_ng_n{}^{-1}) = (e_1, e_2, ... , e_n) = e$
$a^{-1}a = (g_1{}^{-1}, g_2{}^{-1}, ... , g_n{}^{-1})(g_1, g_2, ... , g_n) = (g_1{}^{-1}g_1, g_2{}^{-1}g_2, ... , g_n{}^{-1}g_n) = (e_1, e_2, ... , e_n) = e$
Hence, all elements of D have an inverse.
Hence, D is a group on the group product operation.

## Exercise 20

**Find a subgroup of $\mathbf{Z}_{12} \bigoplus \mathbf{Z}_{18}$ that is isomorphic to $\mathbf{Z}_9 \bigoplus \mathbf{Z}_4$.**
Well, we know that $Z_9 \bigoplus Z_4$ is isomorphic to $Z_{36}$ since 9 and 4 don't share any common factors (by Theorem 8.2).
So, let's just pick two elements with orders 4 and 9.
I think 3 from $Z_{12}$ will work for an order of 4, and 2 from $Z_{18}$ will work for an order 9.
So our generator becomes (3, 2) $\in Z_{12} \bigoplus Z_{18}$, and the group isomorphic to $Z_9 \bigoplus Z_4$ is simply $< (3, 2) >$

## Exercise 22

**Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbf{Z}_{30} \bigoplus \mathbf{Z}_{20}$.**
By Theorem 8.1, the number of elements of order 15 is the number of elements (a, b) $\in Z_{30} \bigoplus Z_{20}$ such that 15 = lcm(|a|, |b|)
Since the orders have to have a LCM of 15, we only have to choose numbers such that the orders are less than 15 and factors of 15.
In other words, we can only choose elements from $Z_{30}$ and $Z_{20}$ with orders of 1, 3, 5, and 15.
So,
from $Z_{30}$: 1:(e), 3:(10, 20), 5:(6, 12, 18, 24), 15:(2, 4, 8, 14, 16, 22, 26, 28)
from $Z_{20}$: 1:(e), 3:(), 5:(4, 8, 12, 16), 15:()
Case:

i) |a| = 15, |b| = 1 - in this case there are 8 * 1 = 8

ii) |a| = 15, |b| = 3 - in this case there are 8 * 0 = 0

iii) |a| = 15, |b| = 5 - in this case there are 8 * 4 = 32

iv) |a| = 15, |b| = 15 - in this case there are 8 * 0 = 0

v) |a| = 5, |b| = 3 - in this case there are 15 * 0 = 0

vi) |a| = 5, |b| = 15 - in this case there are 4 * 0 = 0

vii) |a| = 3, |b| = 5 - in this case there are 2 * 4 = 8

viii) |a| = 3, |b| = 15 - in this case there are 0 * 0 = 0

ix) |a| = 1, |b| = 15 - in this case there are 1 * 0 = 0

So the sum of all of those is 48.
The number of cyclic subgroups of order 15 in $Z_{30} \bigoplus Z_{20}$ is going to be $\frac{48}{\phi(15)} = 6$