Page 132, Exercises 12 and 26

Exercise 12

Let G be a group. Prove that the mapping α (g) = g^{-1} $\forall$ g $\in$ G is an automorphism if and only if G is Abelian.

$\longrightarrow$

Let α (g) = g^{-1} $\forall$ g $\in$ G be an automorphism and let g, h $\in$ G.

Notice:

$$\begin{aligned} \alpha(gh) &= \alpha(g)\alpha(h) \\ (gh)^{-1} &= g^{-1}h^{-1} \\ h^{-1}g^{-1} &= g^{-1}h^{-1} \\ gh^{-1}g^{-1} &= h^{-1} \\ gh^{-1} &= h^{-1}g \\ hgh^{-1} &= g \\ hg &= gh \end{aligned}$$

$\longleftarrow$

Let G be an abelian group and define α (g) = g^{-1} $\forall$ g $\in$ G

Let g, h $\in$ G : h $\neq$ g.

Notice:

α (h) = h^{-1} and α (g) = g^{-1}.

Since each inverse is unique, $h^{-1} \neq g^{-1}$

Hence, α is 1-1.

Let g $\in$ g : g $\neq$ e

G is a group $\Rightarrow \exists$ g^{-1} $\in$ G : gg^{-1} = e.

Thus, α (g^{-1}) = $(g^{-1})^{-1}$ = g

Hence, G is onto.

Let g, h $\in$ G.

Notice: α (gh) = $(gh)^{-1}$ = $h^{-1}g^{-1}$ = $g^{-1}h^{-1}$ = α (g)α (h).

Hence, α is OP.

Since α is 1-1, onto, and OP, α is an automorphism.

Exercise 26

Suppose that ϕ: $Z_{20} \longrightarrow Z_{20}$ is an automorphism and $\phi(5) = 5$. What are the possibilities for ϕ(x)?

Recall: ϕ is an automorphism if it's 1-1, onto, and OP: ϕ(x * y) = ϕ(x) $\cdot$ ϕ(y)

Here are some:

ϕ(x) = x (i.e. mapping the generator, 1, to the "new" generator, 1)

ϕ(x) = −x (i.e. mapping the generator, 1, to −1)

Example:

$\phi(3) + \phi(4) = 17 + 16 = 33 \Rightarrow 13$

$\phi(3 + 4) = \phi(7) = 13$

I think that ϕ(x) = 3x, 7x, 11x, 13x, 17x, and 19x all work as well.