Assigned: Page 54, Exercise 2, 4, 23, 25, 33

Exercise 2

Which of the following binary operations are associative?

a. subtraction of integers - **No.** $(1 - 1) - (-1) \neq (1) - (1 - (-1))$.

b. division of nonzero rationals - **No.** $(2/4)/2 \neq 2/(4/2)$.

c. function composition of polynomials with real coefficients - **Yes**

d. multiplication of 2 x 2 matrices with integer entries - **Yes**

e. exponentiation of integers - **No.** $2^{(3^4)} \neq (2^3)^4$

Exercise 4

Which of the following sets are closed under the given operation?

a. 0, 4, 8, 12 addition mod 16 - **Yes**

	0	4	8	12
0	0	4	8	12
4	4	8	12	0
8	8	12	0	4
12	12	0	4	8

b. 0, 4, 8, 12 addition mod 15 - **No**

	0	4	8	12
0	0	4	8	12
4	4	8	12	1
8	8	12	1	5
12	12	1	5	9

c. 1, 4, 7, 13 multiplication mod 15 - **Yes**

	1	4	7	13
1	1	4	7	13
4	4	1	9	7
7	7	9	4	1
13	13	7	1	4

d. 1, 4, 5, 7 multiplication mod 9 - **No**

	1	4	5	7
1	1	4	5	7
4	4	7	2	1
5	5	2	7	8
7	7	1	8	4

Exercise 23

(Law of Exponents for Abelian Groups)
Let a and b be elements of an Abelian group and let n be any integer.
Show that $(\mathbf{ab})^n = \mathbf{a}^n\mathbf{b}^n$.
Let a, b $\in$ G, an Abelian group, and let n $\in \mathbb{Z}$

$$\begin{aligned}(ab)^n &= ab \times ab \times ab \times ... \times ab \text{ (n times)}\\ &= a \times a \times a \times ... \times a \times b \times b \times b \times ... \times b \text{ (by commutativity)}\\ &= (a)^n(b)^n\end{aligned}$$

Is this also true for non-Abelian groups?
No. Since this requires commutativity to prove.

Exercise 25

Prove that a group G is Abelian iff $(\mathbf{ab})^{-1} = \mathbf{a}^{-1}\mathbf{b}^{-1}$, $\forall$ a, b $\in$ G.
$\longrightarrow$
Let G be an Abelian group, and let a, b $\in$ G.
$(\mathrm{ab})^{-1} = \frac{1}{ab} = \frac{1}{a}\frac{1}{b}$ (by commutativity) $= \mathrm{a}^{-1}\mathrm{b}^{-1}$
$\longleftarrow$
Let a, b $\in$ G and assume that $(\mathrm{ab})^{-1} = \mathrm{a}^{-1}\mathrm{b}^{-1}$, $\forall$ a, b $\in$ G.
Notice that since $(\mathrm{ab})^{-1} = \frac{1}{ab}$ and $\mathrm{a}^{-1}\mathrm{b}^{-1} = \frac{1}{a}\frac{1}{b}$, this implies that $\frac{1}{(ab)} = (\frac{1}{a})(\frac{1}{b})$, $\forall$ a, b $\in$ G
Since the sequence of division and multiplication does not matter, G is commutative, and therefore Abelian.

Exercise 33

Suppose the table below is a group table. Fill in the blank entries.

	e	a	b	c	d
e	e	-	-	-	-
a	-	b	-	-	e
b	-	c	d	e	-
c	-	d	-	a	b
d	-	-	-	-	-

$\longrightarrow$

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c