Due 4/25 (Wednesday):
All (turn in): Chapter 10, page 206, 14-18, 20, 24

Chapter 10

Recall:
A homomorphism ϕ from a group G to a group G′ is a mapping from G into G′ that preserves the group operation; that is, ϕ(ab) = ϕ(a)ϕ(b) for a, b ∈ G.
The kernel of a homomorphism ϕ from a group G to a group G′ with identity e′ is the set {x ∈ G : ϕ(x) = e′ }.

Exercise 14

Explain why the correspondence x $\longrightarrow$ 3x from Z_{12} to Z_{10} is not a homomorphism.

Because ϕ is not OP:
ϕ(3 * 4) = ϕ(12) = ϕ(0) = (3 * (0 mod 12)) mod 10 = e, and
ϕ(3)ϕ(4) = (3 * (3 mod 12) * 3 * (4 mod 12)) mod 10 = (9 * 3 * 4) mod 10 = (108) mod 10 = 8

Exercise 15

Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and Ker ϕ = {0, 10, 20}.
If ϕ(23) = 9, determine all elements that map to 9.

ϕ(ab mod 30) = ϕ(a mod 30)ϕ(b mod 30)
ϕ(23) = 9.
ϕ(0) = ϕ(10) = ϕ(20) = 0
It looks like it's ϕ(x) = 3x:
ϕ(23) = 3 * 23 mod 30 = 69 mod 30 = 9.
Thus,
ϕ(3), ϕ(13), and ϕ(23) all map to 9.

Exercise 16

Prove that there is no homomorphism from $Z_8 \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose ∃ ϕ: $Z_8 \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Because Z_8 is of order 8, and $|Z_2|$ divides 8, there is an element of order 8 in $Z_8 \bigoplus Z_2$, let's call it z_8.
Thus, $z_8 \in Z_8 \bigoplus Z_2$.
Because ϕ is OP, ∃ z ∈ $Z_4 \bigoplus Z_4$ such that $\phi(z_8)$ = z and |z| = 8.
However, there is no element of order 8 in $Z_4 \bigoplus Z_4$. A contradiction.
Hence, no homomorphism exists.

Exercise 17

Prove that there is no homomorphism from $Z_{16} \bigoplus Z_2$ onto $Z_4 \bigoplus Z_4$.

Suppose $\exists$ ϕ: $Z_{16} \bigoplus Z_2 \longrightarrow Z_4 \bigoplus Z_4$, such that ϕ is a homomorphism.
Since G / Ker ϕ is isomorphic to ϕ(G),
|G / Ker ϕ| = |ϕ(G)| = 16.
Since |G| = 32, |Ker ϕ| = 2.
Since Ker $\phi \leq$ G, e $\in$ Ker ϕ.
Since Ker ϕ is of order 2, the other element in Ker ϕ must have order 2, let's call it k.
The only possibilities for k are: (8, 0), (8, 1), (0, 1)
Since |G / Ker ϕ| has order 16, the possibilities for the order of each c $\in$ G / Ker ϕ are factors of 16: 1, 2, 4, 8, and 16.
Case:

i) Ker ϕ = {(0, 0), (8, 0)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 8.

ii) Ker ϕ = {(0, 0), (8, 1)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 16.

iii) Ker ϕ = {(0, 0), (0, 1)}

   Let's look at the coset c = Ker ϕ + (1, 1) $\in$ G / Ker ϕ

   Notice that the order of c is 16.

However, because the homomorphism is onto, there is an isomorphism from G / Ker ϕ to $Z_4 \bigoplus Z_4$.
Thus, for any element g $\in$ G / Ker ϕ of order p, $\exists$ ϕ(g) of order p.
However, the maximum order of $Z_4 \bigoplus Z_4$ is 4. A contradiction.
Hence, there is no homomorphism.

Exercise 18

Can there be a homomorphism from $Z_4 \bigoplus Z_4$ onto Z_8?
Can there be a homomorphism from Z_{16} onto $Z_2 \bigoplus Z_2$? Explain your answers.

Suppose $\exists$ an onto homomorphism ϕ : G $\longrightarrow$ G′ where G = $Z_4 \bigoplus Z_4$ and G′ = Z_8.
Notice: |G| = 16 and |G′ | = 8.
|G / Ker ϕ| = |ϕ(G)| = 8.
Thus, |Ker ϕ| = 2.
So e $\in$ Ker ϕ. The possibilities for the other element in Ker ϕ are: (2, 2), (2, 0), (0, 2).
For each of those possibilities, the order of the coset Ker ϕ + (1, 1) in the quotient group G / Ker ϕ is 2, 4, and 4, respectively. Those are all divisors of |G| and |G′ |, so it appears to work. Yes there is a homomorphism.

For Z_{16} onto $Z_2 \bigoplus Z_2$, I'm going to say no because you're trying to map a cyclic group onto a non-cyclic group, but I don't have time to rigorously prove it before 5pm. Sorry!

Exercise 20

How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?

Let $\phi : Z_{20} \longrightarrow Z_8$
Since ϕ is onto, $|\phi(G)| = 8$.
However, since G is finite, $|\phi(G)|$ divides $|G|$.
Therefore, 8 divides 20. A contradiction.
Therefore, there are no homomorphisms from Z_{20} to Z_8.

Let ϕ be a homomorphism from Z_8 onto Z_8.
Since ϕ is onto, and the groups are the same order, that means ϕ is an isomorphism.
As far as how many of those there are, I'm not sure. 8, I suppose?

Exercise 24

Suppose that ϕ: $Z_{50} \longrightarrow Z_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine ϕ(x).

   If $\phi(7) = 6$, and $\phi(0) = 0$, then I suppose $\phi(x) = 3x$

b. Determine the image of ϕ.

   $\{0, 3, 6, 9, 12\}$

c. Determine the kernel of ϕ.

   $\{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\}$

d. Determine $\phi^{-1}(3)$. That is, determine the set of all elements that map to 3.

   $\{1, 6, 11, 16, 21, 26, 31, 36, 41, 46\}$