Due 4/23:
All (turn in): page 187, 2, 6, 14
Present (me): 10

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Exercise 2

Prove that A_n is normal in S_n.
WTS: $sA_ns^{-1} \subset A_n \ \forall\ s \in S_n$.
Let $s \in S_n$, $a \in A_n$.
Case:

i) $s \notin A_n$

Then s can be expressed as a product of k transpositions, where k is an odd integer.

Since s is odd, this means that s^{-1} is also odd.

Since sas^{-1} is a product of 2k plus an even number of transpositions, sas^{-1} is even.

Therefore, $sas^{-1} \in A_n \ \forall\ a \in A_n$.

Hence, $sA_ns^{-1} \subset A_n \ \forall\ s \notin A_n$ but still in S_n.

ii) $s \in A_n$

Then sas^{-1} is a product of 3k transpositions (where k is an even integer this time).

Thus, sas^{-1} is even.

Hence, $sA_ns^{-1} \subset A_n \ \forall\ s \in A_n$.

Hence, A_n is normal in S_n.

Exercise 6

Let H = { $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$: a, b, d $\in \mathbb{R}$ and ad $\neq 0$ }.
Is H a normal subgroup of GL(2, $\mathbb{R}$)?

WTS: $gHg^{-1} \subset H \ \forall\ g \in GL(2, \mathbb{R})$

Let $g \in GL(2, \mathbb{R})$, $h \in H$.
$ghg^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} h & -f \\ -g & e \end{bmatrix} \frac{1}{eh-fg}$: a, b, ... h $\in \mathbb{R}$, eh − fg $\neq 0$ and ad $\neq 0$.
We want to check if (first)(fourth) in our new matrix (ghg^{-1}) is 0 or not, and if the third entry is 0.
New matrix:
$\begin{bmatrix} e(ah-bg)-fdg & g(ah-bg)-hdg \\ e(be-fa)+fde & g(be-fa)+hde \end{bmatrix}$
This matrix is a member of H if

$$e(be-fa)+fde = ebe-efa+fde = 0$$

and

$$(e(ah-bg)-fdg)(g(be-fa)+hde)-(eah-ebg-fdg)(gbe-gfa+hde) \neq 0$$

I'm guessing no, since the the restrictions are less powerful than the number of options e, b, f, a, d, g, h, etc. There is probably a counter example.

Exercise 10

Let H = {(1), (12)(34)} in A_4.

a. Show that H is not normal in A_4.

   Well, recall that a subgroup H of G is normal iff gH = Hg ∀ g ∈ G.

   So all we need to do is find a g ∈ A_4 such that gH ≠ Hg.

   Notice: (23) ∈ A_4.

   (23)H = {(23)(1), (23)(12)(34)} = {(23), (1342)}

   H(23) = {(1)(23), (12)(34)(23)} = {(23), (1243)}

   {(23), (1342)} ≠ {(23), (1243)}

   Thus, H is not normal in A_4

b. Referring to the multiplication table for A_4 in Table 5.1 on page 105, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6\alpha_9 H = \alpha_7\alpha_{11} H$.

   α_6 = (243), α_7 = (142), α_9 = (132), and α_{11} = (234)

   So, let's look at both:

   $\alpha_6\alpha_9 H \longleftarrow ? \longrightarrow \alpha_7\alpha_{11} H$

   (243)(132)H ⟵ ? ⟶ (142)(234)H

   {(243)(132)(1), (243)(132)(12)(34)} ⟵ ? ⟶ {(142)(234)(1), (142)(234)(12)(34)}

   {(12)(34), (1)} ⟵ ? ⟶ {(14)(23),(13)(24)}

   Nope! Those sets are not equal, so it's not true that $\alpha_6\alpha_9 H = \alpha_7\alpha_{11} H$.

c. Explain why this proves that the left cosets of H do not form a group under coset multiplication.

   Because the order of the permutations results in different output permutation, which means that coset multiplication isn't associative. Therefore, it can't form a group.

Exercise 14

What is the order of the element 14 + ⟨8⟩ in the factor group $Z_{24}/\langle 8\rangle$?
Well,
Z_{24} / ⟨8⟩ = {z⟨8⟩ : z ∈ Z_{24}}
⟨8⟩ = {e, 8, 16}
14 + ⟨8⟩ = {14, 22, 6}
So, first of all, the group Z_{24} / ⟨8⟩ is going to be of order 8 since ⟨8⟩ has 8 left cosets.
Therefore, the order of 14 + ⟨8⟩ cannot be larger than 8.
Let's let k = |14 + ⟨8⟩|.
By LaGrange's theorem, k divides |Z_{24} / ⟨8⟩|, since ⟨14 + ⟨8⟩⟩ will generate a subgroup of Z_{24} / ⟨8⟩.
Since k clearly isn't 1, it can only be 2, 4, or 8.
14 + 14 + ⟨8⟩ = 4 + ⟨8⟩ = {4, 12, 20}. Nope!
14 + 14 + 14 + 14 = 56 + ⟨8⟩ = 8 + ⟨8⟩ = 0 + ⟨8⟩ = ⟨e, 8, 16⟩.
So, k = |14 + ⟨8⟩| = 4.