Given functions $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$, and $\gamma : C \longrightarrow D$, then

1. **$\gamma(\beta\alpha) = (\gamma\beta)\alpha$ (associativity).**

   Let a $\in$ A. Then $(\gamma(\beta\alpha))(a) = \gamma((\beta\alpha)(a)) = \gamma(\beta(\alpha(a)))$.

   On the other hand, $((\gamma\beta)\alpha)(a) = (\gamma\beta)(\alpha(a)) = \gamma(\beta(\alpha(a)))$.

   So, $\gamma(\beta\alpha) = (\gamma\beta)\alpha$.

2. **If α and β are one-to-one, then $\beta\alpha$ is one-to-one.**

   Let α and β be one-to-one.

   Suppose $\beta\alpha$ is not one-to-one.

   Then, $\exists$ c $\in$ C and $a_1, a_2 \in A$ such that $a_1 \neq a_2$, $\beta(\alpha(a_1)) = c$, and $\beta(\alpha(a_2)) = c$.

   Since β is one-to-one, $\beta(\alpha(a_1)) = c$ and $\beta(\alpha(a_2)) = c$ implies $\alpha(a_1) = \alpha(a_2)$.

   Since α is one-to-one, $\alpha(a_1) = \alpha(a_2)$ implies $a_1 = a_2$, a contradiction.

   Hence, $\beta\alpha$ is one-to-one.

3. **If α and β are onto, then $\beta\alpha$ is onto.**

   Let α and β be onto.

   Suppose $\beta\alpha$ is not onto.

   Then $\exists$ c $\in$ C such that $\forall$ a $\in$ A, $\beta(\alpha(a)) \neq c$.

   Since β is onto, $\exists$ b $\in$ B such that $\beta(b) = c$.

   Since α is onto, $\exists$ a $\in$ A such that $\alpha(a) = b$.

   But, $\beta(\alpha(a)) = c$. A contradiction.

   Hence, $\beta\alpha$ is onto.

4. **If α is one-to-one and onto, then there is a function α^{-1} from B onto A such that $(\alpha^{-1}\alpha)(a) = a$, $\forall$ a $\in$ A and $(\alpha\alpha^{-1})(b) = b$, $\forall$ b $\in$ B.**

   Part 1:

   Let α be one-to-one and onto function from A to B.

   Assume $\alpha(a)$ is defined $\forall$ a $\in$ A.

   Let a $\in$ A and let $\alpha(a) = b$.

   Since α is one-to-one, b is only mapped to by a.

   Since α is onto, every element in B is mapped to by an element in A.

   Notice also that every element in B is mapped to only once, since α is one-to-one as well.

   Thus, $\forall$ a $\in$ A, there exists a unique $\alpha(a)$, and for each unique $\alpha(a)$, $\exists$ a unique a.

   Hence, $\exists$ a function α^{-1} such that $(\alpha^{-1}\alpha)(a) = a$, $\forall$ a $\in$ A

   Part 2:

   Let α be one-to-one and onto function from A to B.

   Let b $\in$ B.

   Since α is onto, $\exists$ a $\in$ A such that $\alpha(a) = b$.

   Since α is one-to-one, the only element that maps to b is a.

   Thus, any b $\in$ B can only map backwards to one a $\in$ A, and that a can only map forwards to b.

   Hence, $\exists$ a function α^{-1} such that $(\alpha\alpha^{-1})(b) = b$, $\forall$ b $\in$ B