1. Prove Pascal's Formula $\binom{\alpha}{k} = \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k}$ for any $\alpha \in \mathbb{R}$ and k $\in \mathbb{Z}_{\geq 0}$. (Note: You will need to use the falling factorial definition.)

$$\begin{aligned}
\binom{\alpha}{k} &= \binom{\alpha-1}{k-1} + \binom{\alpha-1}{k} \\
&= \frac{(\alpha-1)!}{((\alpha-1)-(k-1))!(k-1)!} + \frac{(\alpha-1)!}{((\alpha-1)-k)!k!} \\
&= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!}{(\alpha-1-k)!k!} \\
&= \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} + \frac{(\alpha-1)!(\alpha-k)\frac{1}{k}}{(\alpha-k)!(k-1)!} \\
&= \frac{(\alpha-1)! + (\alpha-1)!(\alpha-k)\frac{1}{k}}{(\alpha-k)!(k-1)!} \\
&= \frac{k(\alpha-1)! + (\alpha-1)!(\alpha-k)}{(\alpha-k)!k!} \\
&= \frac{\alpha(\alpha-1)!}{(\alpha-k)!k!} \\
&= \frac{\alpha!}{(\alpha-k)!k!}
\end{aligned}$$

2. Determine the generating function for each of the following sequences:

a. 1, r, r^2, r^3, ...
   $1 + rx + r^2x^2$ ... $\longrightarrow \frac{1}{1-rx}$

b. 1, − 1, 1, − 1, ...
   $1 - x + x^2 - x^3 \longrightarrow \frac{1}{1+x}$

c. $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, -\binom{\alpha}{3}$, ...
   $\binom{\alpha}{0} - \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 - \binom{\alpha}{3}x^3$ ...
   $1 - \alpha\, x + \frac{\alpha(\alpha-1)}{2*1}x^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3*2*1}x^3$ ...
   $1 - \alpha\, x + \frac{[\alpha]_{(2)}}{[2]_{(2)}}x^2 - \frac{[\alpha]_{(3)}}{[3]_{(3)}}x^3$ ...
   $\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}x^k$
   $(1-x)^\alpha$

d. 1, $\frac{1}{1!}$, $\frac{1}{2!}$, $\frac{1}{3!}$, ...
   $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$...
   e^x

e. 1, $\frac{-1}{1!}$, $\frac{1}{2!}$, $\frac{-1}{3!}$, $\frac{1}{4!}$, ...
   $1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3$...
   $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 ... - 2(\frac{1}{1!}x + \frac{1}{3!}x^3 ...)$
   $e^x - \sinh x$

f. $\binom{0}{2}, \binom{1}{2}, \binom{2}{2}, \binom{3}{2}$, ...
   $\binom{0}{2} + \binom{1}{2}x + \binom{2}{2}x^2 + \binom{3}{2}x^3$ ...
   $-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{[2]_{(2)}}x^2 + \frac{[3]_{(2)}}{[2]_{(2)}}x^3$ ...
   $-\frac{1}{2} + 0x + \frac{[2]_{(2)}}{2}x^2 + \frac{[3]_{(2)}}{2}x^3$ ...

Is this the right process? How do you know when to use EGF vs GF?

3. Given the Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$,

   a. Solve the recursion by writing it as a linear homogenous recursion and finding the characteristic polynomial. Write your answer in the form $c_1 {q_1}^n + c_2 {q_2}^n$. (Note: we have already solved this up to finding the constants in class. Finish the problem.)

   $f_n = f_{n-1} + f_{n-2}$
   $0 = f_n$ - $f_{n-1} - f_{n-2}$
   $q^n - q^{n-1} - q^{n-2} = 0$
   $q^{n-2}(q^2 - q^1 - 1) = 0$
   Thus, the solution has the form $f_n = c_1(?)^n\ c_2(?)^n$.
   $q = \frac{1 \pm \sqrt{5}}{2}$
   $f_n = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$
   $f_0 = c_1 + c_2$
   $f_1 = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1$
   Let $f_0 = 0$, $f_1 = 1$. Solving for c_1 and c_2 gives us $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = \frac{-1}{\sqrt{5}}$
   Thus, $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

   b. Solve the recursion by using generating functions. (Note: Use a partial fraction decomposition to finish the problem.)

   $f_n = f_{n-1} + f_{n-2}$
   $h_n = h_{n-1} + h_{n-2}$
   $0 = h_n - h_{n-1} - h_{n-2}$
   Let $g(x) = h_0 + h_1 x^1 + h_2 x^2$ ...
   Then,

$$\begin{aligned} g(x) =& h_0 + h_1x^1 + h_2x^2 ... \\ -xg(x) =& - h_0x^1 - h_1x^2 - h_2x^3 ... \\ -x^2g(x) =& - h_0x^2 - h_1x^3 - h_2x^4 ... \end{aligned}$$

   Thus,

$$(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1 + (h_2 - h_1 - h_0)x^2 + (h_3 - h_2 - h_1)x^3 + ...$$

   But since $0 = h_n - h_{n-1} - h_{n-2}$,

$$(1 - x - x^2)g(x) = h_0 + (h_1 - h_0)x^1$$
$$g(x) = \frac{h_0 + (h_1 - h_0)x}{(1 - x - x^2)}$$

   Plugging in $h_0 = 0$ and $h_1 = 1$,

$$g(x) = \frac{x}{(1 - x - x^2)}$$
$$g(x) = \frac{x}{(1 - x - x^2)}$$
$$g(x) = \frac{x}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$
$$g(x) = \frac{A}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{B}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$

$$g(x) = \frac{1/2}{(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{1/2}{(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$
$$g(x) = \frac{1}{2(x + (-\frac{1}{2} + \frac{\sqrt{5}}{2})} + \frac{1}{2(x - (-\frac{1}{2} + \frac{\sqrt{5}}{2})}$$
$$g(x) = \frac{1}{2}\frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) + x)} - \frac{1}{2}\frac{1}{((-\frac{1}{2} + \frac{\sqrt{5}}{2}) - x)}$$

At this point, I'm not sure how to convert to Power Series

$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

4. Prove that the Fibonacci number f_n is even if, and only if, divisible by 3.

   Wait.. 2 is a fibonacci number that is even and not divisible by 3.. So is 8.

   $\longrightarrow$

   Assume: f_n is even (i.e. $\exists$ t $\in \mathbb{Z}$ such that $f_n = 2t$)

   $\longleftarrow$

   Assume: 3 divides f_n (i.e. $\exists$ t $\in \mathbb{Z}$ such that $f_n = 3t$)

5. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the colors red, white, or blue. Let h_n be the number of colorings in which there is an even number of red squares (the example from class).

   a. Reproduce the exponential generating function solution from class.

   Colors: R, W, B. R is even.

   EGF:

$$(1 + \frac{x^2}{2!} + \frac{x^4}{4!}...)(1 + \frac{x^1}{1!} + \frac{x^2}{2!}...)^2$$
$$\frac{1}{2}(e^x + e^{-x})e^x e^x$$
$$\frac{1}{2}(e^{3x} + e^x)$$
$$\frac{1}{2}(\sum \frac{3^n x^n}{n!} + \sum \frac{x^n}{n!})$$
$$\frac{1}{2}(\sum (3^n + 1)\frac{x^n}{n!})$$
$$\Rightarrow \sum (\frac{3^n + 1}{2})\frac{x^n}{n!}$$

   b. Solve this by using a standard generating function and partial fractions.

   GF:

$$(1 + x^2 + x^4...)(1 + x + x^2...)^2$$
$$\frac{1}{1-x^2}\frac{1}{1-x}\frac{1}{1-x}$$
$$\frac{1}{(1+x)(1-x)^3}$$
$$\frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}$$

   After partial fractions:

$$\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{array}$$

A = $\frac{1}{8}$, B = $\frac{1}{8}$, C = $\frac{1}{4}$, D = $\frac{1}{2}$

$$\Rightarrow \frac{1}{8}\sum(-1)^n x^n + \frac{1}{8}\sum x^n + \frac{1}{4}\sum nx^{n-1} + \frac{1}{2}\sum n(n-1)x^{n-2}$$

How do I get from above to below?

$$\sum(\frac{3^n+1}{2})x^n$$

c. Reproduce the associated recursion for h_n.

d. Using your answer from part c, solve the recursion using the generating function method for non-homogeneous recursions.

6. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the colors red or blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find a recurrence relation that h_n satisfies, then derive a formula for h_n.

   Colors: R, B.

   h_n is the number of colorings such that no two adjacent squares are red.

   In other words, B $\Rightarrow$ R or B and R $\Rightarrow$ B.

   $h_1 = 2$, $h_2 = 3$, $h_3 = 5$, $h_4 = 8$, $h_5 = 13$...

   It looks like the number of ways that color the nth square blue is just h_{n-1}, and the number of ways to color the nth square red is the number of ways that color the (n-1)th square blue, which is just h_{n-2}.

   So it looks like a recurrence relation would be $h_n = h_{n-1} + h_{n-2}$

   Since this is Fibonacci (and we already did this):

   $h_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n + \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

7. Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which apples % 2 = 0, oranges $\leq$ 2, bananas % 3 = 0, and pears $\leq$ 1. Then find a formula for h_n from the generating function.

   GF:

$$(1+x^2+x^4+...)(1+x+x^2)(1+x^3+x^6...)(1+x)$$

$$\frac{1}{(1-x)^2}\frac{1-x^3}{1-x}\frac{1}{(1-x)^3}\frac{1-x^2}{1-x}$$

$$\frac{1}{(1+x)(1-x)}\frac{(1-x)(1+x+x^2)}{1-x}\frac{1}{(1-x)^3}\frac{(1-x)(1+x)}{1-x}$$

$$\frac{(1+x+x^2)}{(1-x)^4}$$

$$\frac{1}{(1-x)^2} + \frac{-3}{(1-x)^3} + \frac{3}{(1-x)^4}$$

$$\sum x^{2n} - 3\sum x^{3n} + 3\sum x^{4n} \longrightarrow \sum x^{2n} - 3x^{3n} + 3x^{4n}$$

8. Determine the exponential generating function for the following sequence:

   a. 0!, 1!, 2!, ...

$$g^{(e)}(x) = \frac{0!}{0!} + \frac{1!}{1!}x + \frac{2!}{2!}x^2...$$
$$g^{(e)}(x) = 1 + x + x^2...$$

   b. $[\alpha]_{(\underline{0})}$, $[\alpha]_{(\underline{1})}$, $[\alpha]_{(\underline{2})}$, $[\alpha]_{(\underline{3})}$, ... (Note: $[\alpha]_{(\underline{n})}$ is the falling factorial.)

$$g^{(e)}(x) = \frac{\alpha}{0!} + \frac{\alpha(\alpha-1)}{1!}x + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}x^2...$$
$$g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{\alpha!}{(\alpha-n-1)!n!}$$

9. Let h_n denote the number of ways to color the square of a 1-by-n board with the colors red, white, blue, and green in such a way that the numbers of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence, then find a simple formula for h_n.

   Colors: RWBG. R is even, W is odd.

   EGF:

$$(1 + \frac{x^2}{2!} + \frac{x^4}{4!}...)(x + \frac{x^3}{3!} + \frac{x^5}{5!}...)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}...)^2$$
$$\frac{1}{2}(e^x + e^{-x}) * \frac{1}{2}(e^x - e^{-x}) * e^x * e^x$$
$$\frac{1}{4}(e^{2x} - e^{-2x}) * e^{2x}$$
$$\frac{1}{4}(e^{4x} - 1)$$

   $f(x) = \frac{1}{4}(e^{4x} - 1) \longrightarrow f'(x) = \frac{1}{4}(4e^{4x}) \longrightarrow f''(x) = \frac{1}{4}16e^{4x} \longrightarrow f'''(x) = \frac{1}{4}64e^{4x} \longrightarrow f^n(x) = \frac{1}{4}(4^n e^{4x})$

   So,

   $h_n = \frac{4^n}{4n!}$ **Which doesn't seem right since n! grows faster than 4^n. Also, where goes - 1?**

10. Determine the number of ways to color the squares of a 1-by-n board using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

    Colors: RGBO. R, G are even.

    GF:

$$(1 + x^2 + x^4...)^2(1 + x + x^2 + x^3...)^2$$
$$\frac{1}{1-x^2}\frac{1}{1-x^2}\frac{1}{1-x}\frac{1}{1-x}$$
$$\frac{1}{(1-x)^4(1+x)^2}$$
$$\frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-x)^4} + \frac{E}{(1+x)} + \frac{F}{(1+x)^2}$$
$$A\sum x^n + E\sum(-1)^n x^n$$

11. Determine the number of n-digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

 d_1, d_2... d_n where d_i % 2 = 1

 1 and 3 each occur a nonzero, even number of times. 5, 7, and 9 can occur any amount of times.

 So:

$$(x^2 + x^4 + x^6 ...)^2(1 + x + x^2 ...)^3$$

 which is

$$(\frac{1}{(1-x)^2} - 1)^2(\frac{1}{1-x})^3$$

$$(\frac{1-(1-x)^2}{(1-x)^2})^2(\frac{1}{1-x})^3$$

$$(\frac{2x-x^2}{(1-x)^2})^2(\frac{1}{1-x})^3$$

$$\frac{(2x-x^2)^2}{(1-x)^7}$$

$$\frac{x^4-4x^3+4x^2}{(1-x)^7}$$

 which decomposes into:

$$\frac{-1}{(x-1)^3} + \frac{2}{(x-1)^5} + \frac{-1}{(x-1)^7}$$

$$-\sum x^{3n} + 2\sum x^{5n} - \sum x^{7n} \longrightarrow \sum -x^{3n} + 2x^{5n} - x^{7n}$$

12. Solve the recurrence relation:

 a. $h_n = 4h_{n-2}$, $h_0 = 0$, $h_1 = 1$, and $n \geq 2$.
 0, 1, 0, 4, 0, 16, 0, 64...
 $h_n - 4h_{n-2} = 0$
 $q^{n-2}(q^2 - 4) = 0$
 $h_n = a(2)^n + b(-2)^n$
 $0 = a + b$ and $1 = 2a - 2b$
 $b = -\frac{1}{4}$, $a = \frac{1}{4}$
 $h_n = \frac{1}{4}2^n - \frac{1}{4}(-2)^n$

 b. $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$. $n \geq 3$.
 $q^{n-3}(q^3 - q^2 - 9q^1 - 9) = 0$
 $(q^2 - 9)(q^1 + 1) = 0$
 $(q - 3)(q + 3)(q + 1) = 0$
 $h_n = a(3)^n + b(-3)^n + c(-1)^n$
 So, $0 = a + b + c$, $1 = 3a - 3b - c$, $2 = 9a + 9b + c$
 Hence, $a = \frac{1}{4}$, $b = 0$, $c = -\frac{1}{4}$
 $h_n = \frac{1}{4}(3)^n + -\frac{1}{4}(-1)^n$

 c. $h_n = 4h_{n-1} + 4^n$, $h_0 = 3$ and $n \geq 1$.
 3, 16, 80, 384...

13. Let h_n = the number of ternary strings of length n made up of 0's, 1's, and 2's, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$h_n = h_{n-1} + 2h_{n-2}$$

with $h_0 = 1$, $h_1 = 3$, and then find a formula for h_n.

14. Compute the Stirling numbers of the first and second kind up to n = 6 using their recursive formulas.

S(6, 6) = 6 S(5, 6) + S(5, 5)

s(6, 6) = s(5, 5) − 5s(5, 6)

15. Prove the Stirling numbers of the second kind satisfy:

Recall: S(p, k) = k S(p − 1, k) + S(p − 1, k − 1) and

$$S(p, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^p$$

a. S(n, 1) = 1

b. S(n, 2) = $2^{n-1} - 1$

c. S(n, n − 1) = $\binom{n}{2}$

16. Prove the Stirling numbers of the first kind satisfy:

Recall: s(p, k) = s(p − 1, k − 1) − (p − 1)s(p − 1, k) and

$$s(p, k) = \sum_{i=k}^{p} p^{i-k} s(p+1, i+1)$$

a. s(n, 1) = (n − 1)!

b. s(n, n − 1) = $\binom{n}{2}$

17. Write $[n]_{(\underline{k})}$ as a polynomial in n for k = 5, 6, 7. (Do not use distribution!)

$[n]_{(\underline{k})}$ = n(n − 1)(n − 2)...(n − k)

$[n]_{(\underline{k})} = \sum_{p=0}^{k}(-1)^{k-p}s(k,p)n^p$

$[n]_{(\underline{5})} = \sum_{p=0}^{5}(-1)^{5-p}s(5,p)n^p$

$[n]_{(\underline{5})} = -s(5,0) + s(5,1)n - s(5,2)n^2 + s(5,3)n^3 - s(5,4)n^4 + s(5,5)n^5$

$[n]_{(\underline{5})} = 4!n - s(5,2)n^2 + s(5,3)n^3 - \binom{5}{2}n^4 + n^5$

s(5, 2) = 4s(4, 2) + 3! and s(5, 3) = $4\binom{4}{2}$ + s(4, 2)

18. Find a closed formula for the sequence: 1, 6, 15, 28, 45, 66, 91, ... (Use a difference table.)

1	6	15	28	45	66	91
	5	9	13	17	21	25
		4	4	4	4	4
			0	0	0	0

$h_n = 1\binom{n}{0} + 5\binom{n}{1} + 4\binom{n}{2}$