Homework Due 10/12/17: (13 problems) Section 4.2 pages 177 - 178; 1, 2, 4, 5(a)(c)(e)(g)(i)(k), 9, 10, 17, 18 (for 5(i) define tn to be 1 over sm, and then show that 1 over sm goes to 0)

Corollary 4.2.5

If $\{t_n\}$ converges to t and $t_n \geq 0 \; \forall \; n \in \mathbb{N}$, then $t \geq 0$

Example 4.2.6

If $\{t_n\}$ converges to t and $t_n \geq 0 \; \forall \; n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \sqrt{t_n} = \sqrt{t}$$

Proof.

Side Note

For $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$|\sqrt{t_n} - \sqrt{t}| < \epsilon \; \forall$ m $\geq$ N

Notice that $\lim_{n\to\infty} t_n = t$, $t \geq 0$
Case **(i)**:

$$\begin{aligned} |\sqrt{t_n} - \sqrt{t}| &= \frac{|(\sqrt{t_n} - \sqrt{t})(\sqrt{t_n} + \sqrt{t})|}{|\sqrt{t_n} + \sqrt{t}|} \\ &= \frac{|t_n - t|}{\sqrt{t_n} + \sqrt{t}} \\ &\leq \frac{|t_n - t|}{\sqrt{t}} \\ &= (\frac{1}{\sqrt{t}})|t_n - t| \end{aligned}$$

For $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$|t_n - t| < \sqrt{t} \times \epsilon$, $\forall$ n $\geq$ N **(2)**

Side Note

$$\sqrt{t} + \sqrt{t_n} \geq \sqrt{t}$$
$$\frac{1}{\sqrt{t} + \sqrt{t_n}} \leq \frac{1}{\sqrt{t}}$$
$$\sqrt{t_n} \geq 0$$
$$\sqrt{t} > 0$$

So, $\sqrt{t_n} + \sqrt{t} > 0 \; \forall \; n \in \mathbb{N}$

From **(1)** and **(2)**,

$|\sqrt{t_n} - \sqrt{t}| \leq \frac{|t_n - t|}{\sqrt{t}} < \frac{\sqrt{t} \times \epsilon}{\sqrt{t}} = \epsilon$, $\forall$ n $\geq$ N

Hence, result in this case.

---Side Note---

If
$|s_n - s| \leq k|a_n| \ \forall \ n \geq N$
and if
$\lim_{n\to\infty} a_n = 0$,
then $\lim_{n\to\infty} s_n = s$

Case **(ii)**: $t = 0$
Then, for $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$t_n = |t_n - 0| < \epsilon^{\,2}$, $\forall$ n $\geq$ N
Thus, $\sqrt{t_n} < \epsilon$, $\forall$ n $\geq$ N
In other words,
$|\sqrt{t_n} - 0| < \epsilon$, $\forall$ n $\geq$ N
So, $\lim_{n\to\infty} \sqrt{t_n} = 0 = \sqrt{t}$
Hence, result.

□

Theorem 4.2.7 - "The Ratio Test"

Suppose that $\{s_n\}$ is a sequence of **positive** terms (i.e. $s_n > 0$, $\forall$ n $\in \mathbb{N}$) and $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = L$.
If $L < 1$, then $\lim_{n\to\infty} s_n = 0$

Proof.

For $\epsilon = \frac{1-L}{2} > 0$,
$\exists$ N $\in \mathbb{N}$ st
$|\frac{s_{n+1}}{s_n} - L| < \frac{1-L}{2}$, $\forall$ n $\geq$ N
So,
$\frac{s_{n+1}}{s_n} = |\frac{s_{n+1}}{s_n}| = |(\frac{s_{n+1}}{s_n} - L) + L| \leq |\frac{s_{n+1}}{s_n} - L| + |L| < (\frac{1-L}{2}) + L = \frac{1+L}{2} = \frac{1}{2} + \frac{L}{2} < \frac{1}{2} + \frac{1}{2}$ (which is 1)
Define $c = \frac{1+L}{2}$
Then,
$s_n \times \frac{s_{n+1}}{s_n} < \subset s_n$, $\forall$ n $\geq$ N where $c < 1$
So, $s_{n+1} < \subset s_n$, $\forall$ n $\geq$ N
Now

$$s_{N+1} < c^1 s_N$$
$$s_{N+2} < cs_{N+1} < c^2 s_N$$
$$s_{N+3} < cs_{N+2} < c^3 s_N, \text{ etc.}$$

So,
$s_{N+K} \leq c^k s_N$, $\forall$ k $\in \mathbb{N} \cup \{0\}$
Thus,
$s_m \leq c^{m-N} s_N$ $\forall$ m $\geq$ N
$s_m \leq c^m \frac{s_N}{c^N}$ $\forall$ m $\geq$ N
$N + k = m$
$k = m - N$

$|s_m - 0| = (\frac{s_N}{c^N})$ **(1)**

Side Note

Theorem 4.1.8
If $|s_m - s| \leq k|a_m|$ and
$\lim_{m\to\infty} a_m = 0$
then $\lim s_m = s$
Also, recall HW 5 7(f): If $|x| < 1$, then $\lim_{n\to\infty} (x^n) = 0$

From **(1)**, it follows by Example 7(f) pg 170 and Theorem 4.1.8, that $\lim_{n\to\infty} s_n = 0$

□

Example 5(g): $s_n = \frac{1-n}{2^n} = \frac{1}{2^n} - \frac{n}{2^n}$ (or, $v_n - u_n$)
Suppose $u_n = \frac{n}{2^n} > 0 \; \forall \; n \in \mathbb{N}$,
$\frac{t_{n+1}}{t_n} = \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} = \frac{1}{2}\frac{n(1+\frac{1}{n}}{n} = \frac{1}{2}\frac{1+\frac{1}{n}}{1} = \frac{1+\frac{1}{n}}{2} = \frac{1}{2} + \frac{1}{2n}$
Which approaches $\frac{1}{2}$ as $n \longrightarrow \infty$

Definition 4.2.9

Infinite Limits:
A sequence $\{s_n\}$ is said to **diverge** to ∞, written as $\lim_{n\to\infty} s_n = \infty$, provided that
$\forall \; M \in \mathbb{R}$, $\exists \; N \in \mathbb{N}$ st
$s_n > M, \forall \; n \geq N$
(i.e. $s_n = (-1)^n$
Similarly, $\{s_n\}$ diverges to $-\infty$, written as $\lim_{n\to\infty} s_n = -\infty$, if, provided that
for every $M \in \mathbb{R}$, $\exists \; N(M) \in \mathbb{N}$ st
$s_n < M, \forall \; n \geq N$

Theorem 4.2.12

Suppose that $\{s_n\}$, $\{t_n\}$ are sequences st $s_n \leq t_n \; \forall \; n \in \mathbb{N}$

a. If $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} t_n = \infty$

b. If $\lim_{n\to\infty} s_n = -\infty$, then $\lim_{n\to\infty} t_n = -\infty$

On the homework, the proof, using the definition, about ”one comment away” from being done.

Theorem 4.2.13

Let: $\{s_n\}$ be a sequence of **positive** numbers
Then $\lim_{n\to\infty} s_n = \infty$
iff $\lim_{n\to\infty} \frac{1}{s_n} = 0$

Proof.

$\longrightarrow$
Suppose that $\lim_{n\to\infty} s_n = \infty$
Want to show: $\lim_{n\to\infty} \frac{1}{s_n} = 0$

Side Note

$\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \epsilon$
(which implies that $s_n > \frac{1}{\epsilon}$)
$\forall$ n $\geq$ N

$\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$s_n > \frac{1}{\epsilon}$, $\forall$ n $\geq$ N
Hence,
$|\frac{1}{s_n} - 0| = \frac{1}{s_n} < \epsilon$, $\forall$ n $\geq$ N
Which shows that $\lim_{n\to\infty} \frac{1}{s_n} = 0$
$\longleftarrow$
Conversely, assume that $\lim_{n\to\infty} \frac{1}{s_n} = 0$
Want to show: $\lim_{n\to\infty} s_n = \infty$

Side Note

For M $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st
$\frac{1}{s_n} < \frac{1}{M}$
$s_n >$ M
$\forall$ n $\geq$ N

Let: M $\in \mathbb{R}$, M $>$ 0
Then $\exists$ N(M) $\in \mathbb{N}$ st
$\frac{1}{s_n} = |\frac{1}{s_n} - 0| < \frac{1}{M}\ \forall$ n $\geq$ N
Hence, $s_n >$ M, $\forall$ n $\geq$ N.
Hence, result.

□