Ch 4: Sequences

4.1: Convergence

Definition 1: Sequence

A **sequence** is a function S: $\mathbb{N} \longrightarrow \mathbb{R}$
We write S(n) = S_n $\forall$ n $\in \mathbb{N}$ and refer to $\{S_n\}$ (the book uses (S_n)) as the **sequence**.
We refer to the set { S_n : n $\in \mathbb{N}$ } as the range of the sequence.
————————————————————Side Note————————————————————-
$S_n = (-1)^n$ $\forall$ n $\in \mathbb{N}$
$\{(-1)^n\}$
range$\{S_n\}$ = $\{-1, 1\}$
Here $\{S_n\}$ = $\{1, -1, 1, -1...\}$

An alternative to writing $\{S_n\}$ for a sequence is to list the elements: S_1, S_2, ... S_n
Sometimes the domain of the sequence is $\mathbb{N} \cup \{0\}$ or $\{n \in \mathbb{N} : n \geq m\}$ for some m $\in \mathbb{N}$.
In this case, we write $\{S_n\}_{n=0}^{\infty}$ or $\{S_n\}_{n=m}^{\infty}$
Note 1: A denumerable set (or a countably infinite set) S is a set for which there is a bijection S: $\mathbb{N} \longrightarrow \mathbb{R}$
This bijection may be thought of as a sequence $\{S_n\}$, where S_n = S(n) $\forall$ n $\in \mathbb{N}$ of distinct terms.

Definition 4.1.2

A sequence $\{S_n\}$ is said to **converge** to s $\in \mathbb{R}$ provided that $\forall$ $\epsilon > 0$
$\exists$ N $\in \mathbb{N} \leq$ n st
$|S_n - S| < \epsilon$
————————————————————Side Note————————————————————-
-—-—-(-—-)-—-—-—-—
s6, s5, sminusep, S / Sn, splusep, s4, s3, s2, s1

We call s the **limit** of the sequence and write:
$\lim_{n\longrightarrow\infty} S_n$ = s or lim S_n or $S_n \longrightarrow$ s as n $\longrightarrow \infty$.
If a sequence does not converge, then it is said to diverge.

Example 4.1.3

Show that the sequence $\{S_n\}$, where $S_n = \frac{1}{n}$ $\forall$ n $\in \mathbb{N}$, $(\{S_n\})$ converges to 0.

Proof.
Want to show: $|\frac{1}{n} - 0| < \epsilon$ for sufficiently large values of n
Now:

$$|\frac{1}{n} - 0| = \frac{1}{n} \tag{1}$$

Since $\frac{1}{n} < \epsilon$ implies n $> \frac{1}{\epsilon}$,
By the AP (Theorem 3.3.10),
$\exists$ N $\in \mathbb{N}$ st N $> \frac{1}{\epsilon}$
Thus,
$\frac{1}{N} < \epsilon$ and $\frac{1}{n} \leq \frac{1}{N} \leq \epsilon$, $\forall$ n $\geq$ N.
From **(1)**, $|\frac{1}{n} - 0| < \epsilon$, $\forall$ n $\geq$ N

[Let N $\in \mathbb{N}$ satisfy N $> \frac{1}{\epsilon}$.
Then $\forall$ n $\geq$ N, $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$]

$\square$

Example 4.1.4

Prove that for $\{\frac{1}{\sqrt{n}}\}$, the limit is 0.

Proof.

Let: $\epsilon > 0$
Then:
$|\frac{1}{\sqrt{n}} - 0| = \frac{1}{\sqrt{n}}$ $\forall$ n $\in \mathbb{N}$ **(1)**

$\frac{1}{\sqrt{n}} < \epsilon$
$\frac{1}{n} < \epsilon$ ^{2}
n $> \frac{1}{\epsilon^2}$
By Theorem 3.3.10 a),
$\exists$ N $\in \mathbb{N}$ st N $> \frac{1}{\epsilon^2}$
From **(1)**,
$|\frac{1}{\sqrt{n}} - 0| = \frac{1}{\sqrt{n}} > \epsilon$, $\forall$ n $\geq$ N

$\square$

Example 4.1.5

Show that if $S_n = 1 + \frac{1}{2^n}$, then $S_n \longrightarrow 1$ as n $\longrightarrow \infty$.

Proof.

Let: $\epsilon > 0$
Then
S_n − S
$|1 + \frac{1}{2^n} - 1| = \frac{1}{2^n} \leq \frac{1}{n} = \frac{1}{N}$ $\forall$ n $\in \mathbb{N}$
Then if N $\in \mathbb{N}$ st $\frac{1}{N} < \epsilon$
Then $|1 + \frac{1}{2^n} - 1| < \epsilon$ $\forall$ n $\geq$ N

$\square$

Theorem 4.1.8

Let: $\{S_n\}$ and $\{a_n\}$ be sequences, s $\in \mathbb{R}$
If some k $>$ 0 and some m $\in \mathbb{N}$, we have:
$|S_n - s| \leq k|a_n|$, $\forall$ n $\geq$ m **(1)**
and if $\lim_{n\longrightarrow\infty} a_n = 0$, then $\lim_{n\longrightarrow\infty} S_n = s$.

Proof.

For $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st
$|a_n| = |a_n - 0| < \frac{\epsilon}{k}$, $\forall$ n $\geq$ N **(2)**
From **(1)**,
$|S_n - s| \leq k|a_n| < k(\frac{\epsilon}{k}) = \epsilon$, $\forall$ n $\geq$ N
Hence, $S_n \longrightarrow$ as n $\longrightarrow \infty$.

□

Example 4.1.11

Prove that if $S_n = n^{\frac{1}{n}}$, $\forall$ n $\in \mathbb{N}$,
then,
$S_n \longrightarrow 1$ as n $\longrightarrow \infty$

Proof.

Recall that
$n^{\frac{1}{n}} = e^{\frac{1}{n} \ln n}$
a^x, $0 < a \in \mathbb{R} = e^{xlna}$, $x \in \mathbb{R}$
Notice that $n^{\frac{1}{n}} \geq 1$, $\forall$ n $\in \mathbb{N}$
We write that:
$n^{\frac{1}{n}} = 1 + b_n$, where $b_n \geq 0$
Thus:
$(n^{\frac{1}{n}})^n = (1 + b_n)^n$
$n = (1 + b_n)^n$
Recall:
$[(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} + ... + \binom{n}{r} a^{n-r}b^r ... + \binom{n}{n-1} ab^{n-1} + \binom{n}{n} a^0b^n]$
where
$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for r = 0, 1, ... n
$\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{1}{2}n(n-1)$
Thus,
$n = (1 + b_n)^n$
$= 1 + nb_n + \frac{1}{2}n(n+1){b_n}^2 + ... + {b_n}^2$ **(1)**
Want to show: $\lim\limits_{n\to\infty} b_n = 0$
From **(1)**,
$n \geq \frac{1}{2}n(n-1){b_n}^2$, $\forall$ n ≥ 2
$1 \geq \frac{1}{2}(n-1){b_n}^2$, $\forall$ n ≥ 2
Then ${b_n}^2 \leq \frac{2}{n-1} < \epsilon$, $\forall$ n $\geq$ N,
where N $\in \mathbb{N}$ is chosen st $N > 2\epsilon^{\,2} + 1$ (FIX?)

${b_n}^2 \leq \frac{2}{n-1} \leq \epsilon^{\,2}$
$\frac{n-1}{2} > \frac{1}{\epsilon^2}$
$n - 1 > \frac{2}{\epsilon^2}$
$n > \frac{2}{\epsilon^2} + 1$
Hence, $b_n < \epsilon$, $\forall$ n $\geq$ N.
This proves that $\lim\limits_{n\to\infty} b_n = 0$, implying that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

□

Example 4.1.12

Prove that the sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$ is divergent.

Proof.
Here $\{S_n\}$ = 0, 2, 0, 2...
We use contradiction.
Suppose: the sequence converges to s $\in \mathbb{R}$

For $\epsilon = 1$, $\exists$ N $\in \mathbb{N}$ st

$$|1 + (-1)^n - s| < 1 \tag{1}$$

$\forall$ n $\geq$ N
Notice that from **(1)**,

$$|s| < 1 \tag{2}$$

$\forall$ odd n $\geq$ N
Also from **(1)**,

$$|2 - s| < 1 \tag{3}$$

$\forall$ even n $\geq$ N
From **(2)**, $-1 <$ s < 1
From **(3)**,
$-1 <$ 2 - s < 1
$-3 < -$s < -1
$3 >$ s > 1
$1 <$ s < 3
It is a contradiction that $-1 <$ s < 1 AND $1 <$ s < 3.
Hence, $\{S_n\}$ diverges.

$\square$