Homework: page 148-149, #1-4, 6, 8

Heine-Borel Theorem

$\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Proof.
$\longrightarrow$
Done.

$\longleftarrow$
Suppose: S is closed and bounded.
Let: S $\subset \bigcup_{\alpha \in I}$ G_α where G_α is open $\forall$ $\alpha \in$ I
Since is is bounded, sup S, inf S $\in \mathbb{R}$ both exist.
Define, for x $\in \mathbb{R}$,
S_x = S $\cap$ $(-\infty$, x].
S $\subset \bigcup_{x \in S}$ N(x, ϵ)
β = { x $\in \mathbb{R}$: S_x has a finite subcover from the G_α's}
$\beta \neq \emptyset$, inf S $\in \beta$
S_{infS} = S $\cap$ $(-\infty$, inf S]
We need to prove that S has a finite subcover of the G_α's.
If β is unbounded above, then $\exists$ z $\in \beta$ st z > sup S.
Then S_z = S $\cap$ $(-\infty$, z] = S
Since S_z = S has a finite subcover of the G_α's, we see that, in this case, S is compact.
We prove that β is unbounded above using contradiction.

Suppose: β is bounded above.
Thus, sup $\beta \in \mathbb{R}$ exists.

Case i): sup $\beta \in$ S.

In this case, $\exists$ $\epsilon \in$ I st sup $\beta \in G_{\alpha_0}$
Since G_{α_0} is open, $\exists$ $\epsilon_0 > 0$ st
N(sup β , ϵ_0) = (sup $\beta - \epsilon_0$, sup $\beta + \epsilon_0$) $\subset G_{\alpha_0}$
By the definition of the supremum,
$\exists$ $x_0 \in \beta$ st
sup $\beta - \epsilon_0 < y_0 \leq$ sup B < sup B + $\frac{\epsilon_0}{2}$ < sup $\beta + \epsilon_0$
Since $x_0 \in \beta_1$, $\exists$ k $\in \mathbb{N}$ and $\{\alpha_1, \alpha_2, ... \alpha_n\} \subset$ I
st $S_{x_0} \subset \bigcup_{i=1}^{k} G_{\alpha_i}$

Side Note

S_{x_0} = S $\cap$ $(-\infty$, x_0]
$S_{sup\beta}$ + $\frac{\epsilon_0}{2}$
= S $\cap$ $(-\infty$, sup β + $\frac{\epsilon_0}{2}$]

This produces the contradiction that (sup β + $\frac{\epsilon_0}{2}$) $\in \beta$

Case ii):

sup $\beta \in \mathbb{R} \setminus$ S, which is open since S is closed.
Thus, $\exists\ \epsilon_1 > 0$ st N(sup β , ϵ_1) $\subset \mathbb{R} \setminus$ S

————————————————Side Note————————————————-
—-(——————-)—-
sup - ep1, sup B, sup B + ep1/2, supB + ep1

As in case i), $\exists$ $x_1 \in \beta$ st
sup $\beta - \epsilon_1 < x_1 \leq$ sup $\beta <$ sup $\beta + \frac{\epsilon_1}{2} <$ sup $\beta + \epsilon_1$
From **(1)**, N(sup β , ϵ_1) = (sup $\beta - \epsilon_1$, sup $\beta + \epsilon_1 \cap$ S = $\emptyset$
—-(——]———-]—-)— supB-ep0, x0inB, supB, supBplusEpOver2, supBplusEp0
Notice that:
S_{x_1} = S $\cap$ $(-\infty$, $x_1]$ = S $\cap$ $(-\infty$, sup $\beta + \frac{\epsilon_1}{2}]$
Again we obtain the contradiction that (sup $\beta + \frac{\epsilon_1}{2}) \in \beta$
Hence, result by contradiction.

□

Theorem 3.5.6: Bolzano - Weierstrass Theorem

If a bounded set S $\subset \mathbb{R}$ contains an infinite number of points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.

Proof.

Suppose: $\exists$ S $\subset \mathbb{R}$ where S has an infinite number of points and S is bounded but S$'$ = $\emptyset$
Since cl S = S $\cup$ S$'$ = S $\cup$ $\emptyset$ = S, we can see by Theorem 3.4.17 a) that S is closed.
Since S is also bounded, it follows by the Heire-Borel theorem that S is compact.
Let: x $\in$ S
Then x $\notin$ S$'$, so $\exists\ \epsilon_x > 0$ st
N(x, ϵ_x) $\cap$ S = {x}

————————————————Side Note————————————————-
——(—-———)——
x-ep(x?), x, yMemS, xplusep(x?)
If x $\in$ S$'$, then:
$\neg[\forall\ \epsilon > 0$, N*$(x, \epsilon)$ $\cap$ S $\neq \emptyset]$
$\exists\ \epsilon > 0$ st N(x, ϵ) $\cap$ S = {x}

Then:
S $\subset \bigcup_{x \in S}$ N(x, ϵ_x)
Since S is compact,
$\exists$ k $\in \mathbb{N}$ and {x_1, x_2, ... x_k} $\subset$ S
S $\subset \bigcup_{i=1}^{k}$ N(x_{i_1}, ϵ_{i_1})
However, S $\cap$ ($\bigcup_{i=1}^{k}$ N(x_{i_1}, ϵ_{i_1})) = {x_1, x_2, ... x_k}
This produces the contradiction that S contains a **finite** number of points.
Hence, S$'$ $\neq \emptyset$

□

Theorem 3.5.7 (F.I.P.)

Let: $\{K_\alpha\}_{\alpha \in I}$ be a family of compact sets, where I is an index.
Suppose that the intersection of any finite subfamily of the K_α's has a nonempty intersection.
Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$

Proof.

Assume that $\bigcap_{\alpha \in I} K_\alpha = \emptyset$
Then $\mathbb{R} \setminus (\bigcap_{\alpha \in I} K_\alpha) = \bigcup_{\alpha \in I} (\mathbb{R} \setminus K_\alpha) = \mathbb{R}$
Notice, by the Heine-Borel Theorem that $\mathbb{R} \setminus K_\alpha$ is open $\forall\ \alpha \in I$.
Let: $\alpha_0 \in I$
Since K_{α_0} is compact,
$\exists$ k $\in \mathbb{N}$ and $\{\alpha_1, \alpha_2, ... \alpha_n\} \subset I$ st.
$K_{\alpha_0} \subset \bigcup_{\alpha \in I} (\mathbb{R} \setminus K_\alpha) \subset \bigcup_{i=1}^{k} (\mathbb{R} \setminus K_{\alpha_0})$

——————————Side Note——————————-

If A $\subset$ B, then $\mathbb{R} \setminus B \subset \mathbb{R} \setminus A$
Let x $\in \mathbb{R} \setminus B$.
Then x $\notin$ B.
So, x $\notin$ A.
Thus, x $\in \mathbb{R} \setminus A$

$\mathbb{R} \setminus (\bigcup_{i=1}^{k}(\mathbb{R} \setminus K_\alpha)) \subset \mathbb{R} \setminus K_{\alpha_0}$
So,
$\bigcap_{i=1}^{k} K_{\alpha_i} \subset \mathbb{R} \setminus K_{\alpha_0}$
We obtain the contradiction that:
$\bigcap_{i=0}^{k} K_{\alpha_i} = \emptyset$
Hence, result. □

Corollary 3.5.8 Nested Intervals Theorem

Let: $\{A_n\}_{n=1}^{\infty}$ be a family of nonempty closed bounded intervals in $\mathbb{R}$ st $A_{n+1} \subset A_n\ \forall$ n $\in \mathbb{N}$
Then:
$\bigcap_{n=1}^{\infty} A_n \neq \emptyset$

Proof.

We use Theorem 3.5.7.
Will this be contradiction?
Suppose: $\forall$ k $\in \mathbb{N}$, that $\{n_1, n_2, ... n_k\} \subset \mathbb{N}$
Then,
$\bigcap_{i=1}^{k} A_{ni} = A_m \neq \emptyset$
where
m = max $\{n_1, n_2, ... n_k\}$

——————————Side Note——————————-

—-[—–[—[—]—]—]—- not imp, not imp, not imp, A3, A2, A1

□