Section 3.5: Compact Sets

Three big areas of analysis: compactedness, continuity, and connectedness.

Definition 3.5.1

A set s $\subset \mathbb{R}$ is said to be compact if every **open cover** has a finite **subcover**
(i.e. if S $\subset \bigcup_{\alpha \in I}$ G$_\alpha$) ,
where G$_\alpha$ is open $\forall$ $\alpha \in$ I; then $\exists$ n $\in \mathbb{N}$ and $\exists$ {n$_1$, n$_2$, ... n$_k$} $\subset$ I
st S $\subset \bigcup_{i=1}^{n} G_{\alpha_i}$

Example 3.5.2

a. Show that S = (0, 2) is not compact.

b. Show that S = {x$_1$, x$_2$, ... x$_n$} $\subset \mathbb{R}$ is compact.

(a)
Notice that:

$$(0, 2) \subset \bigcup_{n=1}^{\infty} (\frac{1}{n}, 3) \tag{1}$$

If (0,2) were compact, then from **(1)** there would exist a **finite** subcover.
Assume: (0, 2) is compact.
So $\exists$ k $\in \mathbb{N}$ and {n$_1$, n$_2$, ... n$_k$} $\subset \mathbb{N}$ $_k$ st

$$(0, 2) \subset \bigcup_{i=1}^{k} (\frac{1}{n_i}, 3) \tag{2}$$

Choose m = max {n$_1$, n$_2$, ... n$_k$}
Then, notice that ($\frac{1}{n_i}$, 3) $\subset$ ($\frac{1}{m}$, 3) $\forall$ i = 1, 2, ... k
From **(1)**, (0, 2) $\subset$ ($\frac{1}{m}$, 3).
Notice that $0 < \frac{1}{m+1} < \frac{1}{m}$
and $\frac{1}{m+1} \in$ (0, 2).
However, $\frac{1}{m+1} \notin$ ($\frac{1}{m}$, 3).

(b)
Suppose that S $\subset \bigcup_\alpha G_\alpha$ $(\alpha \in I)$
where I is an index set and G$_\alpha$ is open $\forall\alpha \in$ I.
$\forall i = 1, 2, ...$ n, $\exists\alpha_i \in$ I st x$_i \in G_{\alpha_i}$
Then, S $\subset \bigcup_{i}^{n} G_{\alpha_i}$
We see that a **finite** subset of $\mathbb{R}$ is compact.

Lemma 3.5.4

If $\emptyset \neq S \subset \mathbb{R}$ and S is **closed** and **bounded**, then S has a maximum and a minimum. In fact, in this, max S = sup S, and min S = inf S.

Proof.

Since S is bounded, inf S, sup S $\in \mathbb{R}$ both exist.
Want to show: max S = sup S

For $\epsilon > 0$, $\exists s_1(\epsilon) \in$ S st
sup S $-\epsilon < S_1 \leq$ sup S $<$ sup S $+\epsilon$.
So,
$-\epsilon < s_1 - supS \leq \epsilon$
Thus, $s_1 \in$ N(sup S, ϵ).
So,

$$\text{N}(\text{sup S}, \epsilon) \ \cap \ \text{S} \neq \emptyset \tag{1}$$

Also, sup S $+\frac{\epsilon}{2} \in$ N(sup S, ϵ) and sup S $+\frac{\epsilon}{2} \in \mathbb{R} \setminus S$.
(s $\leq$ sup S $\forall$ s $\in$ S, and sup S $\in$ S)

$$\text{N}(\text{sup S}, \epsilon) \ \cap \ \mathbb{R} \ \setminus \text{S} \neq \emptyset \tag{2}$$

From **(1)** and **(2)**, sup S $\in$ bd S $\subset$ S, since S is closed. Hence, sup S = max S.

□

Theorem 3.5.5 (Heine-Borel)

A subset $\emptyset \neq$ S $\subset \mathbb{R}$ is compact iff S is closed and bounded.

Proof.
$\longrightarrow$
Suppose: S is compact
Want to show: S_∞ is bounded
Notice that S $\subset \bigcup_{n=1}^{\infty}$ ($-$n, n) = $\mathbb{R}$, **???**
where ($-$n, n) = N(0, n) is open $\forall n \in \mathbb{N}$.
$G_n \subset G_{n+1}$ $\forall n \in \mathbb{N}$.
Since S is compact, $\exists k \in \mathbb{N}$ and {n_1, n_2, ... n_k} $\subset \mathbb{N}$ st
S $\subset$ from i=1 to k $\bigcup$ ($-n_i$, n_i),

Let: m = max {n_1, n_2, ... n_k}.
Then, ($-n_i$, n_i) $\subset$ ($-$m, m) $\forall i = 1, 2, ...k$.
Thus, S $\subset$ ($-$m, m).
So, $|S| <$ m, $\forall s \in$ S.
Or, equivalently,
$-$m $<$ s $<$ m, $\forall$ s $\in$ S.
Hence, S is bounded.

Want to show: S is closed
Suppose: S is not closed
Thus, $\exists$ p $\in$ cl S $\setminus$ S, i.e. p $\in$ S$'$.

Side Note

S is closed iff cl S = S $\cup$ S$'$ = S
S $\subset$ S $\cup$ S$'$
If cl S $\neq$ S, then S $\subset$ S $\cup$ S$'$

Notice that:
$\bigcap_{n=1}^{\infty}$ [p $-\frac{1}{n}$, p $+\frac{1}{n}$] = {p}
So, is equal to $\mathbb{R} \setminus \{p\}$.

S $\subset \mathbb{R} \setminus \bigcap_{n=1}^{\infty}[p - \frac{1}{n}, p + \frac{1}{n}]$
S $\subset \bigcup_{n=1}^{\infty} \mathbb{R} \setminus [p - \frac{1}{n}, p + \frac{1}{n}]$
S $\subset \bigcup_{n=1}^{\infty}$ [($-\infty$, p $-\frac{1}{n}$) $\bigcup$ (p $+\frac{1}{n}$, ∞)]

Since S is compact, $\exists$ k $\in \mathbb{N}$ and {n_1, n_2, ... n_k} $\subset \mathbb{N}$ st
S $\subset \bigcup_{i=1}^{k}$ [($-\infty$, p $-\frac{1}{n_i}$) $\bigcup$ (p $+\frac{1}{n_i}$, ∞)]
Let: m = max {n_1, n_2, ... n_k}
Then ($-\infty$, p $-\frac{1}{n_i}$) $\bigcup$ (p $+\frac{1}{n_i}$, ∞) $\subset$ ($-\infty$, p $-\frac{1}{m}$) $\bigcup$ (p $+\frac{1}{m}$, ∞).
Thus, S $\subset$ [($-\infty$, p $-\frac{1}{m}$) $\bigcup$ (p $+\frac{1}{m}$, ∞)]
$\longleftarrow$
Conversely,
Suppose: S is closed and bounded
Want to show: S is compact
Let: S $\subset \bigcup_{\alpha \in I}^{\infty}$ G_α, where G_α is open $\forall \alpha \in I$ (some index)
$\forall x \in \mathbb{R}$, define:
S_x = S $\cap$ ($-\infty$, x]
Also define the set:
β = {x $\in \mathbb{R}$: S_x is covered by a finite collection of the G_α's}
Notice that S is closed and bounded, so sup S = max S, inf S = min S,
and S_{infS} = S $\cap$ ($-\infty$, inf S] = {inf S} = {min S} (since by Lemma 3.5.4, inf S = min S)

Now, since min S = inf S $\in$ S, then $\exists$ $\alpha_0 \in$ I st inf S $\in$ G_α (α_0).
This proves that
S_{infS} = {inf S} $\subset G_{\alpha_0}$
Hence, inf S $\in \beta \neq \emptyset$.

$\square$