Misc. Notes:

Ex 3.4.8. e)
$\mathbb{R}$ is both open and closed.
int $\mathbb{R} = \mathbb{R}'$
$\emptyset =$ bd $\mathbb{R} \subset \mathbb{R}$
—
S $\subset \mathbb{R}$
$s \in S'$ if, $\forall \epsilon > 0$, N*(x, ϵ) $\cap$ S $\neq \emptyset$
—
HW: pages 141 - 142, numbers 6, 7, 15, 17, 19, 21

Theorem 3.4.17 - pg 118

Let: $S \subset \mathbb{R}$
Then

a. S is closed iff S$'$ $\subset$ S

b. cl S is a closed set

c. S is closed iff S = cl S

d. cl S = S U S$'$ = S $\cup$ bd S

Proof.

a)

S is closed iff S$'$ $\subset$ S
$\longrightarrow$
Suppose: S is closed.
Want to show: S$'$ $\subset$ S
Let: x $\in$ S'
Thus, $\forall \epsilon > 0$

$$\text{N*}(x, \epsilon) \ \cap\ \text{S} \neq \emptyset \tag{1}$$

Want to show: x $\in S$
Assume: x $\notin$ S
Then, from **(1)**,

$$\text{N}(x, \epsilon) \ \cap\ \text{S} \neq \emptyset \tag{2}$$

and

$$\text{N}(x, \epsilon) \ \cap\ \neg\text{S} \neq \emptyset \tag{3}$$

From **(2)** and **(3)**,
x $\in$ bd S $\subset$ S by definition of a closed set. This is a contradiction.
Hence, x $\in$ S.
This proves:

$$S' \subset S$$

$\longleftarrow$
Conversely,
Suppose: $S' \subset S$
Want to show: $\mathbb{R} \setminus S$ is open $\Rightarrow$ S is closed.
Let: $x \in \mathbb{R} \setminus S$
Want to show: $\exists\ \epsilon > 0$ st $N(x, \epsilon) \subset \mathbb{R} \setminus S$
Since $x \notin S$, we see that $x \notin S'$ because $S' \subset S$.
Thus, $\exists \epsilon > 0$ st $N^*(x, \epsilon) \cap S = \emptyset$
Since $x \notin S$, we have:

$$N(x, \epsilon) \cap S = \emptyset \tag{1}$$

Hence, $N(x, \epsilon) \subset \mathbb{R} \setminus S$, which proves that $\mathbb{R} \setminus S$ is open, or, equivalently, that S is closed.
This completes the proof of a).

b)

cl S is a closed set
Recall that cl $S = S \cup S'$.
Want to show: $\mathbb{R} \setminus$ cl S is open $\Rightarrow$ cl S is closed
Let: $x \in$ cl $(\mathbb{R} \setminus S)$ (aka $(S \cup S'$) Compliment)
We must find an $\epsilon > 0$ st $N(x, \epsilon) \subset \mathbb{R} \setminus S \cup S'$
Now $x \notin S$ and $x \notin S'$.
$\exists \epsilon > 0$ st $N^*(x, \epsilon) \cap S = \emptyset$
However, $x \notin S$, so $\exists\ \epsilon > 0$ st

$$N(x, \epsilon) \cap S = \emptyset \tag{1}$$

Equivalently, $N(x, \epsilon) \subset \mathbb{R} \setminus S$
We claim that $N(x, \epsilon) \cap S' = \emptyset$
Since:

$$\neg[x \in S \cup S']$$

$$\neg[x \in S \text{ or } x \in S']$$

$$x \notin S \text{ and } x \notin S'$$

which is equivalent to $N(x, \epsilon) \subset \mathbb{R} \setminus S'$
Let: $y \in N(x, \epsilon)$
By Theorem 2(a), the set $N(x, \epsilon)$ is open.
So $\exists\ \hat{\epsilon} > 0$ st $N(y, \hat{\epsilon}) \subset N(x,\epsilon)$.
In particular, $y \notin N(x, \epsilon)$.

From **(1)**: $N^*(y, \hat{\epsilon}) \cap S = \emptyset$.
So $y \notin S'$ or, equivalently, $y \in \mathbb{R} \setminus S'$.
Thus, $y \in N(x, \epsilon) \Rightarrow y \in \mathbb{R} \setminus S'$
This proves that $N(x, \epsilon) \subset \mathbb{R} \setminus S'$ or, equivalently,

$$N(x, \epsilon) \cap S' = \emptyset \tag{2}$$

From **(1)** and **(2)**, $N(x, \epsilon) \cap (S \cup S') = \emptyset$.

Hence,

$$N(x,\epsilon) \subset (S \cup S')^C = \text{ cl } S^C \tag{3}$$

Thus, **(3)** and **???** prove that cl S^C is open.
Hence, by Theorem 3.4.7, cl S is closed.

c)

S is closed iff S = cl S (= S ∪ S')
$\longrightarrow$
Suppose: S is closed.
Want to show: S = S ∪ S′.
By definition, S ⊂ S ∪ S′.
Want to show: S ∪ S′ ⊂ S
Let x ∈ S
Since cl S = S ∪ S′ , x ∈ S ∪ S'.
If x ∈ S, then we are finished.
If x ∈ S′ \ S

Side Note

Venn Diagram: (S ( )xxS') **???**

Then by a), S' ⊂ S, since S is closed.
Hence, x ∈ S, and we are finished.
$\longleftarrow$
Conversely,
Suppose: S = $S \cup S'$
Want to show: S is closed.
By (b), cl S is closed.
Since, S = $S \cup S'$ = cl S, S is also closed.

d)

cl S = S ∪ S' = S ∪ bd S
Let: x ∈ S ∪ S'
If x ∈ S, then x ∈ S ∪ bd S.
So, S ∪ S ⊂ S ∪ bd S in this case.
If x ∈ S′ \ S, then ∀ $\epsilon > 0$, N(x,ϵ) ∩ S ≠ ∅, which implies x ∈ ℝ \ S and N(x,ϵ) ∩ ℝ \ S ≠ ∅
Thus, x ∈ bd S ⊂ S ∪ bd S.
Hence, S ∪ S' ⊂ S ∪ bd S.

For the reverse conclusion, let x ∈ S ∪ bd S.
If x ∈ S, then x ∈ S ∪ S′. So, in this case, S ∪ bd S ⊂ S ∪ S′ = cl S.
if x ∈ bd S \ S, then, in particular,
$\forall \epsilon > 0$,

$$N * (x,\epsilon) \cap S \neq \emptyset$$

which implies that x ∈ S' ⊂ S ∪ S'.
Hence, S ∪ bd S ⊂ S ∪ S′.
Hence, result.

□