HW 2: page 140-141, #2-5 (Section 3.4)

Theorem 3.3.10

Each of the following is equivalent to the AP:

a. $\forall$ z $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st n > z

b. $\forall$ x > 0, y $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st nx > y

c. $\forall$ x > 0, $\exists$ n $\in \mathbb{N}$ st $0 < \frac{1}{n} < x$

Proof.

We shall prove:

i) AP $\Rightarrow$ a

ii) a $\Rightarrow$ b

iii) b $\Rightarrow$ c

iv) c $\Rightarrow$ AP

In other words, they all imply each other.

a. AP $\Rightarrow$ a

Suppose: a is false.
So, $\forall$ z $\in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$, P(z, n) (st n $\leq$ z) **???**

Side Note

$\neg[\exists x_1 \forall x_2 \text{ st } p(x_1, x_2)] =$
$\forall x_1, \exists x_2 \text{ st } \neg p(x_1, x_2)$

$\exists z_0 \in \mathbb{R}$ st $\forall$ n $\in \mathbb{N}$, n $\leq z_0$
This indicates that the AP is false.
Thus, AP $\Rightarrow$ a.

b. a $\Rightarrow$ b

Assume: a) is true.
Let: $z = \frac{y}{x} \in \mathbb{R}$
By **(a)**, $\exists$ n $\in \mathbb{N}$ st
$n > \frac{y}{x}$
nx > y
Hence, a $\Rightarrow$ b is true.

c. b $\Rightarrow$ c

Assume: b) is true.
$\forall$ x > 0, if y = 1,
we see from **(b)** that $\exists$ n $\in \mathbb{N}$ st nx > 1
Then,
x > $\frac{1}{n}$ > 0.
Hence, b $\Rightarrow$ c.

d. c $\Rightarrow$ AP

Reminder of c: $\forall$ x where $0 < x \in \mathbb{R}$, $\exists$ n $\in \mathbb{N}$ st. $0 < \frac{1}{n} < x$
Suppose: $\mathbb{N}$ is bounded above. (In other words, that the AP is false.
Thus, $\exists$ $z_0 \in \mathbb{R}$ st $0 < n \leq z_0$, $\forall$ n $\in \mathbb{N}$
$0 < n \leq z_0$
$\frac{1}{n} \geq \frac{1}{z_0}$
This contradicts c with $x = \frac{1}{z_0}$ where $0 < \frac{1}{z_0} \in \mathbb{R}$
Hence, result.

□

Theorems 3.3.13 and 3.3.15

Let: x, y $\in \mathbb{R}$ st x < y
Then:

a. $\exists$ r $\in \mathbb{Q}$ st x < r < y

b. $\exists$ z $\in \mathbb{R} \setminus \mathbb{Q}$ st x < z < y

a

Case:
(i): y > 0
$y = 0.a_1a_2...a_n$ i.e. $0.141 = \frac{141}{1000}$
(ii): $y \leq 0$
$-y \geq 0$, $-y < -x$, $0 \leq -y < -x$
By case **(i)**, $\exists$ r $\in \mathbb{Q}$ st
$-y < r < -x$
$y > -r > x$
$x < -r < y$

b

$\exists$ z $\in \mathbb{R} \setminus \mathbb{Q}$ st x < z < y
Apply **(a)** to $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$ to find r $\in \mathbb{Q}$ st
$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$
$x < r\sqrt{2} < y$
Let: $r\sqrt{2} = z$
$x < z < y$

Hence, result.

Section 3.4: Topology of $\mathbb{R}$

Definitions 3.4.1 and 3.4.2

Let x $\in \mathbb{R}$ and $\epsilon > 0$.

(a)
An ϵ -neighborhood of x is:
N(x, ϵ) = {y $\in \mathbb{R}$: $|y - x| < \epsilon$ }

(b)
A deleted ϵ -neighborhood of x is:
N*(x, ϵ) = {y $\in \mathbb{R}$: $0 < |y - x| < \epsilon$ }

Open Set Topology: Definition 3.4.3 (interior / boundary point)

Let: S $\subset \mathbb{R}$
A point x $\in \mathbb{R}$ is an **interior point** of S if $\exists\ \epsilon > 0$ st N(x, ϵ) $\subset$ S.

If, $\forall\ \epsilon > 0$,
N(x, ϵ) $\cap$ S $\neq \emptyset$
and
N(x, ϵ) $\cap$ $\mathbb{R}$ $\setminus$ S $\neq \emptyset$
Then x is a **boundary point** of S.
The set of all interior points is denoted by **int S**.
The set of all boundary points is denoted by **bd S**.
Nota Bene (N.B.):
int S $\subset$ S and bd S = bd ($\mathbb{R}$ $\setminus$ S)

Side Note

Let: x $\in$ int S
Then $\exists\ \epsilon > 0$ st N(x, ϵ) $\subset$ S
In particular, x $\in$ S. Thus, int S $\subset$ S.
Let: $S^C = \mathbb{R} \setminus S$, and $\mathbb{R} \setminus S^C = S$
Then s $\in$ bd S^C if $\forall\ \epsilon > 0$,
N(x, ϵ) $\cap$ $S^C \neq \emptyset$
N(x, ϵ) $\cap$ $\mathbb{R} \setminus S^C \neq \emptyset$
Thus, N(x, ϵ) $\cap$ ($\mathbb{R}$ $\setminus$ S) $\neq \emptyset$, and N(x, ϵ) $\cap$ S $\neq \emptyset$
So, x $\in$ bd S

Theorem 1

Let: $x \in S \subset \mathbb{R}$
Then either $x \in$ int S, or $x \in$ bd S.

Proof.

Let: $x \in S \subset \mathbb{R}$

i) $\exists\, \epsilon > 0$ st $N(x, \epsilon) \subset S$. Then, by def, $x \in$ int S

ii) $\forall\, \epsilon > 0$, $N(x, \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.
However, since $x \in S$, then $N(x, \epsilon) \cap S \neq \emptyset$.
By definition, $x \in$ bd S.

Hence, result.

□

Section 3.4.4 Examples

a. **Let:** S = (0, 5)
Here, int S = (0, 5) and bd S = {0, 5}
To see this*****,
Let: $x \in (0, 5)$, $\epsilon = \min\{x, 5-x\}$
Then $N(x, \epsilon) \subset (0, 5)$
To see this, let $y \in N(x, \epsilon)$.
Want to show: $0 < y < 5$
Since $y \in N(x, \epsilon)$, we have

$$x - \epsilon < y < x + \epsilon \tag{1}$$

Notice that

$$\epsilon \leq x \tag{2}$$

and

$$\epsilon \leq 5 - x \tag{3}$$

From **(3)**,

$$x + \epsilon \leq x + (5 - x) = 5 \tag{4}$$

From **(2)**,

$$x - x \leq x - \epsilon,\ 0 \leq x - \epsilon \tag{5}$$

From **(1)**, **(4)**, **(5)**,
$0 \leq x - \epsilon < y \leq x + \epsilon < 5$,
We see that $y \in (0, 5)$.
Hence, $0 < y < 5$.
Since $N(x, \epsilon) \subset (0, 5)$, we see that
int S = (0, 5) = S

Want to show: $0 \in$ bd S

Let: $0 < \epsilon < 5$

Notice that:

N(0, ϵ) $\cap$ (0, 5) $\neq \emptyset$ and N(0, ϵ) $\cap$ ($\mathbb{R} \setminus$ (0, 5)) $\neq \emptyset$

Using y = (+/−) $\frac{\epsilon}{2}$, notice that:

y $\in$ N(0, ϵ) since $|(+/-)\frac{\epsilon}{2}| < \epsilon$

2

b. **Let:** S = [0, 5]

Here, int S = (0, 5), bd S = {0, 5}

Notice that bd S $\subset$ S

c. **Let:** S = [0, 5)

Here, int S = (0, 5), bd S = 0, 5

Notice that some bd points are in S, but some aren't.

d. **Let:** S = [2, ∞)

Here, int S = (2, ∞), bd S = {2}

e. **Let:** S = $\mathbb{R}$

int S = $\mathbb{R}$ = S, bd S = $\emptyset$

Here, bd S $\subset$ S

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——————————Side Note——————————-

—-(—-(——————)———)——

0, x-ep, y, x, xplEp, 5

from 0 to x is x, from x to 5 is 5-x

2

——————————Side Note——————————-

——(————(————)—-)—-

0-ep, -ep/2, 0, ep/2, 0plEp, 5
