

Theorem 1 (infinum definition)

Let: $\emptyset \neq S \subset \mathbb{R}$, S is bounded below.

Then S possesses a greatest lower bound denoted by **inf S** (the **infinum** of S), where $\inf S \in \mathbb{R}$, satisfying:

- i) $\inf S \leq s \forall s \in S$
- ii) $\forall \epsilon > 0, \exists s_1$ st $\inf S + \epsilon > s_1$

Proof.

Let: S be bounded below

Then

$$\exists m \in \mathbb{R} \text{ st } m \leq s \forall s \in S \quad (1)$$

Define the set $-S$ to be $\{-s : s \in S\}$

So, $\emptyset \neq -S \subset \mathbb{R}$

From (1), $-S \leq -m \forall s \in S$.

Thus, $-m$ is an upper bound for $-S$.

By the Axiom of Completeness of \mathbb{R} (AoC), $\sup(-S) \in \mathbb{R}$ exists.

By definition,

$$-s \leq \sup(-S), \forall s \in S \quad (2)$$

and $\forall \epsilon > 0, \exists -s_1 \in S$ st

$$\sup(-S) - \epsilon < -s_1 \text{ where } s_1 \in S \quad (3)$$

From (2),

$$-\sup(-S) \leq S \forall s \in S \quad (4)$$

Want to show: $-\sup(-S) = \inf S$

From (3),

$$-\sup(-S) + \epsilon > s_1 \text{ where } s_1 \in S \quad (5)$$

We see that from (4) and (5),

$\inf S = -\sup(-S)$.

Hence, result. □

Theorem 3.3.7

Given nonempty subsets of \mathbb{R} , A, B ($A, B \subset \mathbb{R}$),

Let: $C = \{x + y : x \in A, y \in B\}$

If A and B have suprema, then C has a supremum: $\sup C = \sup A + \sup B$

Proof.

Let: $c \in C$

Then $c = x + y$ for some $x \in A, y \in B$.

It follows that:

$$x \leq \sup A, y \leq \sup B$$

$$x + y \leq \sup A + \sup B$$

$$c \leq \sup A + \sup B$$

$$c \leq \sup A + \sup B \quad \forall c \in C \tag{1}$$

By the AoC, $\sup C \in \mathbb{R}$ exists.

For $\epsilon > 0$, $\exists x_0 \in A, y_0 \in B$ st

$$\sup A - \frac{\epsilon}{2} < x_0 \tag{2}$$

$$\sup B - \frac{\epsilon}{2} < y_0 \tag{3}$$

From (2) and (3),

$$\sup A - \frac{\epsilon}{2} + \sup B - \frac{\epsilon}{2} < x_0 + y_0 = c_0 \in C$$

So,

$$\sup A + \sup B - \epsilon < c_0 \tag{4}$$

From (1) and (4), $\sup C = \sup A + \sup B$

Hence, result. □

Theorem 3.3.8

Suppose $\emptyset \neq D \subset \mathbb{R}$ and

$f : D \rightarrow \mathbb{R}$

$g : D \rightarrow \mathbb{R}$

$f(D) = \{f(x) : x \in D\}$

If $\forall x, y \in D, f(x) \leq g(y)$, then

$f(D)$ is bounded above and $g(D)$ is bounded below.

Furthermore, $\sup(f(D)) \leq \inf(g(D))$

Proof.

Let: $y_0 \in D$

Then $f(x) \leq g(y_0) \forall x \in D$

So, $f(D)$ is bounded above by $g(y_0)$.

By the AoC, $\sup(f(D))$ exists and $\sup(f(D)) \leq g(y_0)$

Since $y_0 \in D$ was arbitrary, we see that

$\sup(f(D)) \leq g(y) \forall y \in D$

Thus, $\sup(f(D))$ is a lower bound for $g(D)$

$g(D) = \{g(y) : y \in D\}$

Hence, $\inf(g(D)) \in \mathbb{R}$ exists and

$\sup(f(D)) \leq \inf(g(D))$

Hence, result. □

Theorem 3.3.9: Archimedean Property / Principle of \mathbb{R} (AP)

The set $\mathbb{N} = \{1, 2, 3, \dots\}$ is unbounded above in \mathbb{R}

Proof.

Suppose: \mathbb{N} is bounded above.

By the AoC, $\sup \mathbb{N} \in \mathbb{R}$ exists.

So,

$$\text{i) } n \leq \sup \mathbb{N} \forall n \in \mathbb{N} \quad \mathbf{(1)}$$

$$\text{ii) } \forall \epsilon > 0, \exists n \in \mathbb{N} \text{ st } \sup \mathbb{N} - \epsilon < n_0 \quad \mathbf{(2)}$$

Using **(2)** with $\epsilon = 1$, $\exists n_0(1) \in \mathbb{N}$ st $\sup \mathbb{N} - \epsilon < n_0$

Then, $\sup \mathbb{N} < 1 + n_0$ **(3)**

See that **(3)** contradicts **(1)** with $n = 1 + n_0 \in \mathbb{N}$

By contradiction, \mathbb{N} is unbounded. □