6.1 Continued

Theorem 6.1.3

Let: I be an interval containing the point c
Assume: f : I $\longrightarrow \mathbb{R}$
f is differentiable at c
iff
for every sequence $\{x_n\}$ in I st $x_n \longrightarrow$ c as n $\longrightarrow \infty$ with $x_n \neq$ c $\forall$ n $\in \mathbb{N}$,
the sequence

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges.
Furthermore, if f is differentiable at c, then the sequence of quotients converges to f(c).
(this looks weird, haven't we already proved it?
they're just saying it to be specific, emphasizing the p iff q and not p iff not q part)

Proof.

$\longrightarrow$
Assume: f′ (c) exists
Then $\lim_{x \to c} (\frac{f(x)-f(c)}{x-c})$ = f′ (c).
By Theorem 5.1.8, if $\{x_n\}$ lies in I, $x_n \neq$ c $\forall$ n $\in \mathbb{N}$, and $x_n \longrightarrow$ c as n $\longrightarrow \infty$, then

$$\lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c)$$

$\longleftarrow$
Conversely,
Assume: $\{x_n\}$ lies in I, $x_n \neq$ c $\forall$ n $\in \mathbb{N}$, and $x_n \longrightarrow$ c as n $\longrightarrow \infty$
Then,

$$\lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c}$$

exists.
Then, by the negation of Theorem 5.1.10,

$$\frac{f(x) - f(c)}{x - c}$$

has a limit at x = c.
Hence, result.

□

Example 6.1.4

a. **Let:** f(x) = |x| = x if x ≥ 0, −x if x < 0. Prove that f is not differentiable at x = 0.

Solution:

Let: $x_n = \frac{(-1)^n}{n}$ ∀ n ∈ ℕ

Notice that if

$$\lim_{x\to 0} \frac{f(x_n) - f(0)}{x_n - 0}$$

exists, then f is differentiable and

$$\lim_{n\to \infty} \frac{f(x_n) - f(0)}{x_n - 0}$$

exists.

Also, notice that $x_n \neq 0$ ∀ n ∈ ℕ , $x_n \longrightarrow 0$ as n $\longrightarrow \infty$ and

$$s_n = \frac{f(x_n) - f(0)}{x_n - 0} = \frac{|\frac{(-1)^n}{n}| - |0|}{\frac{(-1)^n}{n}} = \frac{\frac{1}{n}}{\frac{(-1)^n}{n}} = \frac{1}{(-1)^n}$$

Since s_n = −1, 1, −1, 1, −1, 1, ..., $\{s_n\}$ does not converge.

So by Theorem 6.1.3, f is not differentiable at x = 0.

b. **Let:** f(x) = $3x^2 + 1$ if x < 1, $2x^3 + 2$ if x ≥ 1. Prove that f is differentiable at x = 1.

Solution: We must prove that

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

exists.

We know that:

$\lim_{x\to c}$ g(x) exists iff $\lim_{x\to c^-}$ g(x) = $\lim_{x\to c^+}$ g(x) = L

Left hand side limit:

$$\lim_{x\to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1-} \frac{3x^2 + 1 - 4}{x - 1} = \lim_{x\to 1-} \frac{3(x-1)(x+1)}{x - 1} = \lim_{x\to 1-} 3(x+1) = 6$$

Right hand side limit:

$$\lim_{x\to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1+} \frac{2x^3 + 2 - 4}{x - 1} = \lim_{x\to 1+} \frac{2(x^3 - 1)}{x - 1} = \lim_{x\to 1+} \frac{2(x-1)(x^2 + x + 1)}{x - 1} = \lim_{x\to 1+} 2(x^2+x+1) = 6$$

Hence,

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = 6$$

Practice 6.1.5

Let: f : $\mathbb{R} \longrightarrow \mathbb{R}$ be defined by f(x) = x sin($\frac{1}{x}$) if x $\neq$ 0 and 0 if x = 0
Solution:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x\sin\frac{1}{x}-0)}{x-0}$$

Let: $x_n = \frac{2}{n\pi}\ \forall\ n \in \mathbb{N}$
Then $x_n \neq 0\ \forall\ n \in \mathbb{N}$ and $x_n \longrightarrow 0$ as $n \longrightarrow \infty$.
However,

$$\frac{f(x_n)-f(0)}{x_n-0} = \sin\frac{1}{x_n} = \sin\frac{n\pi}{2}$$

Since sin($\frac{n\pi}{2}$) = 1, 0, −1, 0, 1, 0, −1, ...,
by Theorem 6.1.3, f is not differentiable at x = 0.

Theorem 6.1.6

If f : I $\longrightarrow \mathbb{R}$ is differentiable at a point c $\in$ I, then f is continuous.

Proof.

Recall:
$\lim_{x\to c}$ f(x) = f(c) is saying 3 things:
The limit exists, the function is defined at c, and that they're equal to each other.
(you can't say undefined = undefined)
Let: x $\in$ I with x $\neq$ c
Then

$$f(x) = (\frac{f(x)-f(c)}{x-c})(x-c) + f(c) \longrightarrow f(c)$$

as x $\longrightarrow$ c.
by Theorem 5.1.13.
Thus, by Theorem 5.2.21(d)(a), f is continuous.

□

Theorem 6.1.7

Assume: f : I $\longrightarrow \mathbb{R}$ and g : I $\longrightarrow \mathbb{R}$ are differentiable at c $\in$ I.
Then,

a. If k $\in \mathbb{R}$, then (kf)′ (c) = k * f′ (c)

b. (f + g)′ (c) = f′ (c) + g′ (c)

c. (fg)′ (c) = f(c)g′ (c) + f′ (c)g(c)

d. If g(c) $\neq$ 0, then $(\frac{f}{g})'$ (c) = $\frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}$

There's no homework this week, but make sure you know how to prove a, b, and c on your own.

Proof.

(d):
Let: x $\in$ I, x $\neq$ c
Then,

$$\frac{(\frac{f}{g})(x) - (\frac{f}{g})(c)}{x-c} = \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} = \frac{f(x)g(c) - f(c)g(x)}{[g(x)g(c)](x-c)}$$

We have a problem. What if g(x) − g(c) = 0?
Since g(c) $\neq$ 0 and g is differentiable at c, we know that g is continuous at c by 5.2 HW problem 11 on 213.
Recall:

$$|a| - |b| \leq ||a| - |b|| \leq |a-b|$$

$$|b| \geq |a| - |a-b|$$

so,

$$|g(x)| \geq g(c) - |g(c)| - g(x)| > \frac{|g(c)|}{2} > 0$$

Hence,
$\exists\ \delta > 0$ st

$$-|g(c) - g(x)| > \frac{-|g(c)|}{2}$$

where |x − c| < δ
So we know that:
$\exists$ an interval J $\subset$ I st c $\in$ J and if x $\in$ J, then g(x) $\neq$ 0.
Thus,

$$(\frac{f}{g})(x) - (\frac{f}{g})(c) = \frac{f(x)g(c) - f(c)g(c) \times f(c)g(c) - f(c)g(x)}{g(x)g(c)(x-c)} = \frac{g(c)[f(x)-f(c)]}{g(x)g(c)(x-c)} - \frac{f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}$$

$$= \frac{g(c)f'(c) - f(c)(g'(c)}{[g(c)]^2}$$

□