HW 11: page 220 - 221, #1, 2, 5 and page 226-227, # 1 - 3, 4(a)(b), 5, 11

5.4 Continued

Theorem 5.4.6

f : D $\longrightarrow \mathbb{R}$ is continuous and D is compact.
Then f is uniformly continuous on D.

Proof.

Let: c $\in$ D and let $\epsilon > 0$
Since if is continuous on D, $\exists$ δ (c) > 0 st

$$|f(x) - f(c)| < \frac{\epsilon}{2} \tag{1}$$

whenever |x $-$ c| < δ (c) and x $\in$ D
Notice that:

$$D \subset \bigcup_{c \in D} N(c, \frac{\delta(c)}{2})$$

Since D is compact,

$$D \subset \bigcup_{i=1}^{n} N(c_i, \frac{\delta(c_i)}{2}) \tag{2}$$

Let: $\delta = \min\{\frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, ... \frac{\delta(c_n)}{2}\}$ and x, y $\in$ D st |x $-$ y| < δ
Then, from **(2)**,
$\exists$ k $\in$ {1, 2, ... n} st x $\in$ N(c_k, $\frac{\delta(c_k)}{2}$)
Thus,

$$|x - c_k| < \frac{\delta(c_k)}{2} < \delta(c_k) \tag{3}$$

and

$$|y - c_k| < |y - x| + |x - c_k| < \delta + \frac{\delta(c_k)}{2} \leq \frac{\delta(c_k)}{2} + \frac{\delta(c_k)}{2} = \delta(c_k) \tag{4}$$

Now:

$$|f(x) - f(y)| \leq |f(x) - f(c_k)| + |f(c_k) - f(y)|$$

So, from **(1)**, **(3)**, and **(4)**,

$$|f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, result. □

Practice 5.4.7

Find a continuous function f : D $\longrightarrow \mathbb{R}$ and a Cauchy sequence $\{x_n\}$ in D st $\{f(x_n)\}$ is divergent.
This is a good practice problem because it will show us why the next theorem is so useful.

Proof.

f : (0, 1) $\longrightarrow \mathbb{R}$ where $f(x) = \frac{1}{x}$ (we could have also used x^2)
Let: $x_n = \frac{1}{n} \; \forall \; n \in \mathbb{N}$
Then $\lim\limits_{n\to\infty} x_n = 0$
So $\{x_n\}$ is a Cauchy sequence.
However,
$f(x_n) = \frac{1}{\frac{1}{n}} = n \longrightarrow \infty$ as $n \longrightarrow \infty$
Hence,
$\{f(x_n)\}$ diverges.

□

Theorem 5.4.8

Let: f : D $\longrightarrow \mathbb{R}$ be uniformly continuous on D
Assume: $\{x_n\}$ is a Cauchy sequence in D
Then,
$\{f(x_n)\}$ is a Cauchy sequence.

Proof.

For $\epsilon > 0$, $\exists \; \delta > 0$ st

$$|x - y| < \delta \text{ and } x, y \in D \Rightarrow |f(x) - f(y)| < \epsilon$$

Since $|x_n|$ is Cauchy,
$\exists$ N $\in \mathbb{N}$ st $|x_n - x_m| < \delta$ whenever n, m $\geq$ N
Hence,
$|f(x_n) - f(x_m)| < \epsilon$ whenever n, m $\geq$ N
which shows that $\{f(x_n)\}$ is Cauchy. □

Theorem 5.4.9

A function f : (a, b) $\longrightarrow \mathbb{R}$ is uniformly continuous on (a, b)
iff f can be extended to a function $\widetilde{f}$: [a, b] $\longrightarrow \mathbb{R}$
where
$\widetilde{f}$ is continuous on [a, b].

—Side Note—

We say that a function $\widetilde{f}$: E $\longrightarrow \mathbb{R}$ is an extension of a function f : D $\longrightarrow \mathbb{R}$
if D $\subset$ E and $\widetilde{f}$(x) = f(x), $\forall$ x $\in$ D

Proof.

$\longrightarrow$
Assume: f : (a, b) $\longrightarrow$ is uniformly continuous on (a, b)
Let: $\{x_n\}$ and$\{y_n\}$ be sequences in (a, b) st $x_n \longrightarrow$ a and $y_n \longrightarrow$ b as n $\longrightarrow \infty$
Then $\{x_n\}$ and$\{y_n\}$ are Cauchy sequences in D.
By Theorem 5.4.8, $\{f(x_n)\}$ and $\{f(y_n)\}$ are Cauchy sequences which, therefore, converge.
Let: $\lim_{n\to\infty} f(x_n) = p$ and $\lim_{n\to\infty} f(y_n) = q$
Define $\widetilde{f}$: [a, b] $\longrightarrow \mathbb{R}$ by
$\widetilde{f}$(x) = f(x) if x $\in$ (a, b), p if x = a, and q if x = b
Then $\widetilde{f}$ is an extension of f, which is continuous.
Notice that $x_n \longrightarrow$ a as n $\longrightarrow \infty$ and $\lim_{n\to\infty} \widetilde{f}(x_n) = \lim_{n\to\infty} f(x_n) = p = \widetilde{f}(a)$
Hence,
$\widetilde{f}$(x) is continuous at x = a by Theorem 5.2.2(b).
Similarly,
$\lim_{n\to\infty} \widetilde{f}(y_n) = \lim_{n\to\infty} f(y_n) = q = \widetilde{f}(b)$
Hence,
$\widetilde{f}$(x) is continuous at x = b by Theorem 5.2.2(b).
Since $\widetilde{f}$(x) = f(x) $\forall$ x $\in$ (a, b), then $\widetilde{f}$ is continuous on (a, b).
Hence,
$\widetilde{f}$ is continuous on [a, b].
$\longleftarrow$
Conversely,
Assume: f can be extended to a function $\widetilde{f}$: [a, b] $\longrightarrow \mathbb{R}$ where $\widetilde{f}$ is continuous on [a, b]
By Theorem 5.4.6, $\widetilde{f}$ is uniformly continuous on [a, b], since by the Heine-Borel Theorem, [a, b] is compact.
Hence,
$\widetilde{f}$ is uniformly continuous on (a, b).
Since $\widetilde{f}$(x) = f(x) $\forall$ x $\in$ (a, b),
f is uniformly continuous on (a, b).
Hence, result.

□

Practice: 5.4.10

Use Thm 5.4.9 to determine whether or not the function f(x) = $\sin(\frac{1}{x})$ is uniformly continuous on $(0, \frac{1}{\pi})$.

Proof.

Let: $x_n = \frac{2}{n\pi}$, $\forall$ n $\in \mathbb{N}$
Then $f(x_n) = \sin(n\frac{\pi}{2})$ $\forall$ n $\in \mathbb{N}$
Here, $\lim_{n\to\infty} x_n = 0$
Notice that $\lim_{k\to\infty} f(x_{2k}) = 0$
However, $\lim_{k\to\infty} f(x_{4k-3}) = 1$
Hence,
$\{f(x_n)\}$ does not converge (since, if it did, then all its subsequences would have to converge to the same limit, which they do not).

□

Chapter 6: Section 6.1

Definition 6.1.1

Let: f be real-valued and defined on an interval containing the point c (possibly an end-point)
We say that f is **differentiable at c** (i.e. has a derivative at c) if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists (and is, therefore, finite).
In this case,

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Alternatively,

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x}$$

Example 6.1.2

Let: f(x) = x^2, $\forall$ x $\in \mathbb{R}$
For any c $\in \mathbb{R}$, we have:

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{x^2-c^2}{x-c} = \lim_{x\to c}\frac{(x-c)(x+c)}{x-c} = \lim_{x\to c}(x+c) = 2c$$