

Part of HW 11: 1, 2, and 5 from section 5.3

Hint: 5^x and x^4 are polynomials, so you can assume $g(x)$ is continuous.

Look at 5.3.8 if you have a chance.

Theorem 5.3.2

$D \subset \mathbb{R}$ is compact and $f : D \rightarrow \mathbb{R}$ is continuous $\Rightarrow f(D)$ is compact (i.e. D is compact $\Rightarrow f(D)$ is compact)

Proof.

A set is compact iff every open cover of a set has a finite subcover.

Assume: $f(D) \subset \bigcup_{\alpha \in I} G_\alpha$ and G_α is open $\forall \alpha \in I$

Thus, $D \subset \bigcup_{\alpha \in I} f^{-1}(G_\alpha)$

(i.e. $\bigcup_{\alpha \in I} f^{-1}(G_\alpha) = \bigcup_{\alpha \in I} (H_\alpha \cap D) \subset \bigcup_{\alpha \in I} H_\alpha$)

Side Note

Let: $x \in D$

Then $f(x) \in f(D)$

So, $\exists \alpha_0 \in I$ st $f(x) \in G_{\alpha_0}$

and $x \in f^{-1}(G_{\alpha_0}) \subset \bigcup_{\alpha \in I} f^{-1}(G_\alpha)$

From

$$f^{-1}(G_\alpha) = H_\alpha \cap D, \text{ where } H_\alpha \text{ is open (Theorem 5.2.14)}$$

Since D is compact,

$$D \subset \bigcup_{i=1}^n H_{\alpha_i}$$

Thus,

$$D \subset \bigcup_{i=1}^n (H_{\alpha_i} \cap D) = \bigcup_{i=1}^n f^{-1}(G_{\alpha_i})$$

$$f(D) \subset \bigcup_{i=1}^n G_{\alpha_i}$$

Hence, $f(D)$ is compact.

Side Note

Let: $x \in D$

Then $x \in f^{-1}(G_{\alpha_k})$ where $k \in \{1, 2, \dots, n\}$

So $f(x) \in G_{\alpha_k} \subset \bigcup_{i=1}^n G_{\alpha_i}$

and

$$f(D) \subset \bigcup_{i=1}^n G_{\alpha_i}$$

□

Corollary 5.3.3

$f : D \rightarrow \mathbb{R}$ is continuous and D is compact.

Then $\exists x_1, x_2 \in D$ st

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in D$$

where $f(x_2)$ is the max of $f(D)$ and $f(x_1)$ is the min.

Proof.

$f(D)$ is compact, so by Heine-Borel, $f(D)$ is closed and bounded.

Thus, $\sup f(D)$ exists.

Now, $\sup f(D)$ is an accumulation point of $f(D)$.

Since $f(D)$ is closed,

$\sup f(D) \in f(D)$ and $\sup f(D) = f(x_2)$ for some $x_2 \in D$

i.e. $y \leq \sup f(D) \forall y \in f(D)$

and

$\exists y_n \in f(D)$ st $\sup f(D) - \frac{1}{n} < y_n \leq \sup f(D) < \sup f(D) + \frac{1}{n}$

□

Section 5.4: Uniform Continuity

Definition 5.4.1

Let: $f : D \rightarrow \mathbb{R}$

We say that f is uniformly continuous on D if for $\epsilon > 0$, $\exists \delta > 0$ st $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $x, y \in D$

Recall the old definition of continuity:

f is continuous at $c \in D$ iff $\forall \epsilon > 0$, $\exists \delta > 0$ st $|x - c| < \delta$ and $x \in D \Rightarrow |f(x) - f(c)| < \epsilon$

The old one depends on c and ϵ , where as the new definition only depends on ϵ

Notice that if f is uniformly continuous on D , then f is certainly continuous on D .

Example 5.4.2

Prove that $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 2x$ is uniformly continuous on \mathbb{R}

Proof.

Let: $x, y \in \mathbb{R}$ and $\epsilon > 0$

$|f(x) - f(y)| = |2x - 2y| = 2|x - y| < \epsilon$

whenever $|x - y| < \delta = \frac{\epsilon}{2}$

□

Practice 5.4.3

Negating the definition:

$\exists \epsilon > 0$ st $\forall \delta > 0$, $\exists x, y \in D$ st $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$

in logic, but is another word for and (where but tells you something different is coming)

Example 5.4.4**Let:** $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x^2$ Prove that f is not uniformly continuous on \mathbb{R} *Proof.***Let:** $\epsilon = 1$, $x \in \mathbb{R}^+$ and choose $\delta > 0$ **Let:** $y = x + \frac{\delta}{2}$

Then

$$|x - y| = \frac{\delta}{2} < \delta$$

and

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| = |x + y|\frac{\delta}{2} \quad (1)$$

Let: $x = \frac{1}{\delta}$

Then

$$|x + y| = \left| \frac{1}{\delta} + \frac{1}{\delta} + \frac{\delta}{2} \right| = \frac{2}{\delta} + \frac{\delta}{2} > \frac{2}{\delta} \quad (2)$$

From (1) and (2),

$$|f(x) - f(y)| > \frac{2}{\delta} \frac{\delta}{2} = 1$$

Hence,

 f is NOT uniformly continuous on \mathbb{R}

□

Example 5.4.5**Let:** $f : [-5, 5] \rightarrow \mathbb{R}$ where $f(x) = x^2$ Show that f is uniformly continuous on $[-5, 5]$ *Proof.***Let:** $\epsilon > 0$ and $x, y \in D$

Then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y|(|x| + |y|) \leq |x - y|10 < \epsilon$$

whenever $|x - y| < \delta = \frac{\epsilon}{10}$

□

Theorem 5.4.6

Assume: $f : D \rightarrow \mathbb{R}$ is continuous on a compact set D .

Then, f is uniformly continuous on D

Proof.

Let: $\epsilon > 0$ and $c \in D$

Then $\exists \delta(c) > 0$ st

$$f(x) - f(c) < \frac{\epsilon}{2}$$

whenever $|x - c| < \delta(c)$ and $x \in D$

(this is just the definition of continuity)

Thus,

$$D \subset \bigcup_{c \in D} N(c, \frac{\delta(c)}{2})$$

Since D is compact and $N(c, \delta(c))$ is an open set $\forall c \in D$ (which we proved a long time ago),

$\exists n \in \mathbb{N}$ st

$$D \subset \bigcup_{i=1}^n N(c_i, \frac{\delta(c_i)}{2})$$

Let: $\delta = \min \{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \dots, \frac{\delta(c_n)}{2} \}$

Now let $x, y \in D$ with $|x - y| < \delta$

Then $x \in N(c_k, \frac{\delta(c_k)}{2})$ for some $k \in \{1, 2, \dots, n\}$

Then $|x - c_k| < \frac{\delta(c_k)}{2} < \delta(c_k)$

From **(1)**,

$$|f(x) - f(c_k)| < \frac{\epsilon}{2} \quad \textbf{(2)}$$

Also, $|y - c_k| = |(y - x) + (x - c_k)| \leq |y - x| + |x - c_k| < \delta + \frac{\delta(c_k)}{2} \leq \frac{\delta(c_k)}{2} + \frac{\delta(c_k)}{2} = \delta(c_k)$

From **(1)**,

$$|f(y) - f(c_k)| < \frac{\epsilon}{2} \quad \textbf{(3)}$$

Hence, from **(2)** and **(3)**,

$$|f(x) - f(y)| < |f(x) - f(c_k)| + |f(c_k) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, f is uniformly continuous. □