Section 5.2 Continued

Theorem 5.2.14

A function f : D $\longrightarrow \mathbb{R}$ is continuous on D iff for every open set G in $\mathbb{R}$ $\exists$ an open set H in $\mathbb{R}$ st H $\cap$ D = f^{-1}(G)

Proof.

$\longrightarrow$
Let: G be an open subset of $\mathbb{R}$
Assume: f is continuous on D
If c $\in$ f^{-1}(G), then f(c) $\in$ G.
Since G is open, $\exists$ a neighborhood V of f(c) such that v $\subset$ G
By Theorem 5.2.2(c), $\exists$ a neighborhood U(c) of c, such that
f(U(c) $\cap$ D) $\subset$ V
Now, let H = $\bigcup_{c\in f^{-1}(G)}$ U(G)
Since each neighborhood U(c) is open, it follows that H is open and that H $\cap$ D = f^{-1}(G)
$\longleftarrow$
Let: V be a neighborhood of f(c) since c $\in$ D
Since V is an open set, our hypothesis implies that $\exists$ an open set H $\subset$ $\mathbb{R}$ st H $\cap$ D = f^{-1}(V)
Since f(c) $\in$ V, we have c $\in$ H
But, H is an open set, so $\exists$ a neighborhood U of c st U $\subset$ H.
Thus, f(U $\cap$ D) $\subset$ f(H $\cap$ D) $\subset$ V
From Theorem 5.2.2, f is continuous on D.

□

Corollary 5.2.15

A function f : $\mathbb{R}$ $\longrightarrow$ $\mathbb{R}$ is continuous iff f^{-1}(G) is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$

Example 5.2.16

Define f : $\mathbb{R}$ $\longrightarrow$ $\mathbb{R}$
f(x) = {x if x $\leq$ 2, 4 if x > 2}
If G = (1, 3), then f^{-1}(G) = (1, 2]

Section 5.3: Properties of Continuous Functions

Definition 5.3.1: Boundedness of Function

A function f: D $\longrightarrow$ $\mathbb{R}$ is said to be bounded if the range f(D) is a bounded subset of $\mathbb{R}$ (i.e. f is bounded if there exists M $\in$ $\mathbb{R}$ such that |f(x)| $\leq$ M for all x $\in$ D).
Note: A continuous function may not be bounded even when the domain is bounded.

Theorem 5.3.2

If D $\subset$ $\mathbb{R}$ is compact and f : D $\longrightarrow$ $\mathbb{R}$ is continuous, then
f(D) is compact

Proof.

Let: J = $\{G_\alpha\}$ be an open cover of f(D)
Want to show: J has a finite subcover.
Since f is continuous on D,
Theorem 5.2.14 implies that for each open set G_α in J, $\exists$ an open set H_α st $H_\alpha \cap D = f^{-1}(G_\alpha)$
Moreover, since $f(D) \subset \bigcup G_\alpha$,
it follows that $D \subset \bigcup f^{-1}(G_\alpha) \subset \bigcup H_\alpha$,
Thus, the collection $\{H_\alpha\}$ is an open cover of D.
Since D is compact, $\exists$ finitely many sets H_{α_1}, H_{α_2}, .. H_{α_n} such that
$D \subset H_{\alpha_1} \cup H_{\alpha_2}$ ... H_{α_n}
But then $D \subset (H_{\alpha_1} \cap D) \cup (H_{\alpha_2} \cap D)$ ... $(H_{\alpha_n} \cap D)$
$f(D) \subset G_{\alpha_1} \cup G_{\alpha_2}$ ... G_{α_n}
Therefore,
$\{G_{\alpha_1}, ... G_{\alpha_n}\}$ is a finite subcover of J for f(D)
Therefore, f(D) is compact.

□

Corollary 5.3.5

Let: $D \subset \mathbb{R}$ be compact
$f : D \longrightarrow \mathbb{R}$ is continuous implies f assumes min and max values on D.
That is to say: $\exists$ points $x_1, x_2 \in D$ such that $f(x_1) \leq f(x) \leq f(x_2)\ \forall\ x \in D$

Proof.

By Theorem 5.3.2, f(D) is compact.
From Lemma 3.5.4, f(D) has both a minimum, y_1, and a maximum, y_2.
Since $y_1, y_2 \in f(D)$, there exists $x_1, x_2 \in D$ st $f(x_1) = y_1$ and $f(x_2) = y_2$
Thus,
$f(x_1) \leq f(x) \leq f(x_2)\ \forall\ x \in D$

□

Lemma 5.3.5

Let: $f : [a, b] \longrightarrow \mathbb{R}$ be continuous
$f(a) < 0 < f(b) \Rightarrow \exists\ c \in (a, b)$ st $f(c) = 0$

Proof.

Let: $c = \max\{x : f(x) \leq 0\}$ and $S = \{x \in [a, b] : f(x) \leq 0\}$
Since $a \in S$, S is nonempty.
Notice that S is bounded above by b, so c = sup S exists as a real number in [a, b]
Want to show: f(c) = 0
Suppose: f(c) < 0
Then $\exists$ a neighborhood U of c such that $f(x) < 0$ for all $x \in U \cap [a, b]$
(This comes from Exercise 5.2.13)
Now $c \neq b$, since $f(c) < 0 < f(b)$
Thus, U contains an in between point p st $c < p < b$
But $f(p) < 0$ since $p \in U$

Therefore, p ∈ S
This contradicts c being an upper bound for S.
Suppose: f(c) > 0
Similarly,
If f(c) > 0, then ∃ a neighborhood U of c such that f(x) > 0 for all x ∈ U ∩ [a, b]
Now, c ≠ a, since f(c) > 0 > f(a)
Thus, U contains a point p st a < p < c
Since f(x) > 0 ∀ x ∈ U, no points of S are in [p, c]
That is to say, p is an upper bound for S.
This contradicts c being the least upper bound (supremum) of S.
Hence, f(c) = 0
Since f(a) < 0 < f(b) and f(c) = 0,
∃ c ∈ (a, b) □

Theorem 5.3.6 - Intermediate Value Theorem

Assume: f : [a, b] ⟶ ℝ is continuous
Then f has the intermediate value property on [a, b].
That is, if k is any value between f(a) and f(b),
i.e. f(a) < k < f(b) or f(b) < k < f(a),
then ∃ c ∈ (a, b) st f(c) = k

Proof.

Let: k be between f(a) and f(b)
If f(a) < f(b), from Lemma 5.3.5, consider the continuous function:
g : [a, b] ⟶ ℝ given by g(x) = f(x) − k
Then,
g(a) = f(a) − k < 0
and
g(b) = f(b) − k > 0
From Lemma 5.3.5, ∃ c ∈ (a, b)
st
g(c) = 0 = f(c) − k ⇒ f(c) = k
Similarly, we can prove when f(a) > f(b)

□

Exercise 5.3.7

Using the intermediate value theorem, we can show that every positive number has a positive nth root.
Assume: k > 0, n ∈ ℕ
Let: f(x) = x^n
Notice that f(0) = 0 < k
if b = k + 1, then from Bernolli's inequality (Exercise 3.1.24),

$$b^n = (k+1)^n \geq 1 + kn > k$$

$$f(b) = b^n = (k+1)^n \geq 1 + kn > k$$

Since f is continuous,
∃ c ∈ (0, b) st f(c) = k = c^n, where c is the nth root of k

Theorem 5.3.10

Let: I be a compact interval
Assume: f : I $\longrightarrow \mathbb{R}$ is a continuous function
Then, the set f(I) is a compact interval.

Proof.

From Corollary 5.3.3,
$\exists$ x_1, $x_2 \in$ I st $f(x_1) \leq f(x) \leq f(x_2)$, for all $x \in I$
Let: $m_1 = f(x_1)$, $m_2 = f(x_2)$, and $f(I) \subset \subset [m_1, m_2]$
If $m_1 = m_2$, then $f(I) = \{m_1\} = [m_1, m_2]$, and we're done.
If $m_1 < m_2$ and $k \in (m_1, m_2)$, then by Theorem 5.3.6, we have
$k = f(c)$, $c \in (x_1, x_2)$ and $(m_1, m_2) \subset f(I)$.
m_1, $m_2 \in f(I)$, $[m_1, m_2] \subset f(I)$, f(I) is the compact interval $[m_1, m_2]$, and we are done.

$\square$