HW 10: page 212 - 214, #1, 2 (omit d), 3, 5 (prove the result), 10, 11 (just prove the "max" result), 13, 16 (First prove that for any H $\subset \mathbb{R}$, $f^{-1}(R \setminus H) = \mathbb{R} \setminus f^{-1}(H)$, use this in conjunction with Theorem 5.2.14)

Lec 21 Continued

Theorem 5.2.2

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D
Then the following are equivalent:

a. f is continuous at c

b. If $\{x_n\}$ is any sequence in D st $x_n \longrightarrow c$ as n $\longrightarrow \infty$ (x_n can actually be c),
then $\lim_{n\to\infty} f(x_n) = f(c)$

c. For every neighborhood V of f(c), $\exists$ a neighborhood U of c st $f(U \cap D) \subset V$
Furthermore, if $c \in D'$, then the above are all equivalent to d

d. f has a limit at c and $\lim_{x\to c} f(x) = f(c)$

Proof.

Case:

i) $c \in D \setminus D'$ (i.e. c is an isolated point)
Thus, $\exists$ a neighborhood $U \subset \mathbb{R}$ of c st

$$U \cap D = \{c\}$$

(i.e. U = (c − δ , c + δ) = {c})

(a)

Want to show: f is continuous at x = c

For $\epsilon > 0$, $\exists\ \delta > 0$ st (c − δ , c + δ) $\subset$ U.

This follows since a neighborhood is open. Thus,

$$|f(x) - f(c)| = 0 < \epsilon \text{ whenever} |x - c| < \delta \text{ and } x \in D$$

This means by definition that f(x) is continuous at x = c.

(b)

Let: $\{x_n\} \subset D$ st $x_n \longrightarrow c$ as n $\longrightarrow \infty$

and

For $\epsilon > 0$, $\exists\ \delta > 0$ st (c − δ , c + δ) $\subset$ U

Want to show: $\lim_{n\to\infty} f(x_n) = f(c)$.

Since U is open, $\exists$ N $\in \mathbb{N}$ st

$$|x_n - c| < \delta \text{ for } n \geq N$$

Thus, $x_n \in U$ for $n \geq N$

We see that

$$|f(x_n) - f(c)| = 0 < \epsilon \text{ for } n \geq N$$

Hence, $\lim_{n\to\infty} f(x_n) = f(c)$

(c)

Now,

Let: V be a neighborhood of f(c)

Then, using U as defined prior to (a):

$$f(U \cap D) \subset V$$

Hence, a, b, and c, are equivalent if $c \in D \setminus D'$

ii) $c \in D \cap D'$ (i.e. c is an accumulation point)

(a) is equivalent to **(d)** by Definition 5.2.1

(b) is equivalent to **(d)** by Theorem 5.2.2

(c) is equivalent to **(d)** by Theorem 5.1.8

N.B.: In case **(i)**, we proved that a function is always continuous at an isolated point in its domain.

Sometimes in calculus one, we tell a student that a function is continuous in the interval [A, B] if you can trace it on the chalkboard without having to take your hand off. This is a white lie.

It turns out that as long as it's defined at all points on its domain, it's continuous (i.e. sequences are continuous).

N.B.: In definition 5.2.1, we defined continuity of f at a **point** c in the domain D of f. If $S \subset D$ and f is continuous at each point of S, then f is continuous on S. If f is continuous at all points of D, then f is a continuous function on D.

□

Example 5.2.3

Let: $p : \mathbb{R} \longrightarrow \mathbb{R}$ be a polynomial
In Example 5.1.14, we saw that $\lim_{x\to c} p(x) = p(c)$.
By **(d)** iff **(a)**, from Theorem 5.2.2, we see that p is a continuous function on $\mathbb{R}$

Example 5.2.5

Define $f(x) = x \sin(\frac{1}{x})$, $x \neq 0$, but 0 if $x = 0$
Then f: $\mathbb{R} \longrightarrow \mathbb{R}$
Prove that f is continuous at $x = 0$

We're thinking that the limit of this function at 0 is 0, so,
Let: $\epsilon > 0$
Now,

$$|f(x) - f(0)| = |x \sin\frac{1}{x} - 0| = |x||\sin\frac{1}{x}| \leq |x| * 1 = |x| < \epsilon$$

whenever $0 < |x| < \delta = \epsilon$ (but since we have 0 in there, it's also true whenever $|x| < \delta = \epsilon$) and $x \in D$
Hence, by Definition 5.2.1, $\lim_{x\to 0} f(x) = f(0)$, i.e. f is continuous at $x = 0$

Theorem 5.2.6

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D
Then,
f is **discontinuous** at c iff $\exists$ a sequence $\{x_n\}$ in D st $x_n \longrightarrow$ c but $\lim_{n\to\infty} f(x_n)$ is not f(c).

Proof.

This is not **(a)** iff not **(b)** in Theorem 5.2.2

□

Example 5.2.7

Let: f : $\mathbb{R} \longrightarrow \mathbb{R}$, where f(x) = $\frac{1}{x}$ if x $\neq$ 0, k if x = 0
Prove that f is discontinuous at x = 0.
So,
for $\epsilon > 0$, $\exists\ \delta > 0$ st

$$|f(x) - f(0)| = |\frac{1}{x} - k| < \epsilon$$

whenever $|x - 0| < \delta$ and x $\in$ D
Let: $x_n = \frac{1}{n}\ \forall$ n $\in \mathbb{N}$
Then,
$\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$
So,
$\lim_{n\to\infty} f(x) \neq f(0) = k \in \mathbb{R}$
Note: If we define D = $(-\infty, 0) \cup (0, \infty)$ and let f : D $\longrightarrow \mathbb{R}$ be defined by f(x) = $\frac{1}{x}$,
then f is continuous on D.
If c $\in$ D, then $\lim_{x\to c} f(x) = \frac{1}{c}$ = f(c).
Since c $\in$ D′ , it follows from Theorem 5.2.2 that f is continuous at c.

5.2.8

The Dirichlet function is f: $\mathbb{R} \longrightarrow \mathbb{R}$ defined by:
f(x) = 1 if x $\in \mathbb{Q}$, 0 if x $\in \mathbb{R} \setminus \mathbb{Q}$
Prove that f is discontinuous everywhere.
For $\epsilon = \frac{1}{4}$, we must find $\delta > 0$ st $|f(x) - f(c)| < \frac{1}{4}$ whenever $0 < |x - c| < \delta$ and x $\in$ D
Solution:
Let: c $\in \mathbb{R}$ Case:

i) c $\in \mathbb{Q}$

   Let: $x_n = c + \frac{\sqrt{2}}{n}$

   Then $x_n \in \mathbb{R} \setminus \mathbb{Q}$, $\forall$ n $\in \mathbb{N}$

   and $\lim_{n\to\infty} x_n = c$

   $\lim_{n\to\infty} f(x_n) = 0 \neq f(c) = 1$

   By Theorem 5.2.6, f is not continuous at x = c

ii) c $\in \mathbb{R} \setminus \mathbb{Q}$

   Let: $x_n \in \mathbb{Q}\ \forall$ n $\in \mathbb{N}$ st $x_n \longrightarrow$ c as n $\longrightarrow \infty$

   Then $\lim_{n\to\infty} f(x_n) = 1 \neq f(c) = 0$

Take a look at 5.2.9, but it won't be discussed in this class.

Theorem 5.2.10

Let: f, g : D $\longrightarrow \mathbb{R}$ and c $\in$ D
Assume: f and g are continuous at c

a. f + g and fg are continuous at c

b. $\frac{f}{g}$ is continuous at c provided that g(c) $\neq$ 0

Proof.

(a): Similar to b.
(b):
Let: {x_n} be a sequence in D st $x_n \longrightarrow$ c as n $\longrightarrow \infty$
Then,

$$\lim_{n\to\infty}(\frac{f}{g})(x_n) = \lim_{n\to\infty}\frac{f(x_n)}{g(x_n)} = \frac{\lim_{n\to\infty} f(x_n)}{\lim_{n\to\infty} g(x_n)} = \frac{f(c)}{g(c)} = (\frac{f}{g})(c)$$

By Theorem 5.2.2, **(a)** iff **(b)**, $\frac{f}{g}$ is continuous at c. □

Example 5.2.11

See Exercise #11, page 214
Prove:
(max {f, g})(x) = max {f(x), g(x)} = $\frac{1}{2}$(f(x) + g(x)) + $\frac{1}{2}$|f(x) − g(x)| $\forall$ x $\in$ D
Use 2 cases.

Theorem 5.2.12

Let: f : D $\longrightarrow \mathbb{R}$, g : E $\longrightarrow \mathbb{R}$ where f(D) $\subset$ E.
If f is continuous at c $\in$ D and g is continuous at f(c) $\in$ E, then the composition of f and g given by g o f is continuous at x = c
(This is essentially saying the composition of two continuous functions is also continuous.)

Proof.

Let: W be a neighborhood of g(f(c)).
Since g is continuous at f(c), $\exists$ a neighborhood V of f(c) st g(V $\cap$ E) $\subset$ W by Theorem 5.2.2. (a) iff (c) **(1)**
Since f is continuous at c, there is a neighborhood U of c st f(U $\cap$ D) $\subset$ V **(2)**
Now, f(D) $\subset$ E, so f(U $\cap$ D) $\subset$ E.
Thus,
(2) implies f(U $\cap$ D) $\subset$ V $\cap$ E
So g(f((U $\cap$ D))) $\subset$ W (i.e. (g o f)(U $\cap$ D) $\subset$ W)
By Theorem 5.2.2, g o f is continuous at x = c.

□

Example 5.2.13

q(x) = x sin($\frac{1}{x}$) is continuous at any c $\in \mathbb{R}$ st c $\neq$ 0

Proof.
q(x) = [h(g o f)](x), where f(x) = $\frac{1}{x}$, g(x) = sin x, h(x) = x
Since f, g, and h are all continuous, so is q(x). ☐

Theorem 5.2.14 (Links Topology with Analysis)

A function f : D $\longrightarrow \mathbb{R}$ is continuous on D iff
for every open set G $\subset \mathbb{R}$, $\exists$ an open set H $\subset \mathbb{R}$ st $f^{-1}(G) = H \cap D$
(This is a way of talking about continuity without using distance (i.e. using open sets instead))