Do 20 if you finish 19 on the HW

## Theorem 5.1.13 (as seen in Lec 20)

Let f : D $\longrightarrow \mathbb{R}$ , g : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
If $\lim_{x \to c}$ f(x) = L and $\lim_{x \to c}$ g(x) = M, then

a. $\lim_{x \to c}$ (f + g) = L + M

b. Let k $\in \mathbb{R}$ , $\lim_{x \to c}$ kf = kL

c. $\lim_{x \to c}$ (fg) = LM

d. $\lim_{x \to c}$ $(\frac{f}{g}) = \frac{L}{M}$, provided that M $\neq$ 0

*Proof.*
**(a)** through **(c)** are similar to **(d)**.
**(d)**: Let $\{s_n\}$ be a sequence in D st $s_n \neq$ c $\forall$ n $\in \mathbb{N}$ and $\lim_{n \to \infty} s_n$ = c.
Then, by Theorem 5.1.8, $\lim_{n \to \infty}$ f($s_n$) = L.
Now, $\lim_{n \to \infty}$ g(x) = M $\neq$ 0.
So $\exists$ N $\in \mathbb{N}$ st

$$g(s_n) \neq 0 \text{ for } n \geq N$$

(ask why? next time)
Then, $\lim_{n \to \infty} (\frac{f}{g})(s_n) = \lim_{n \to \infty} \frac{f(s_n)}{g(s_n)} = \frac{\lim_{n \to \infty} f(s_n)}{\lim_{n \to \infty} g(s_n)}$ (by Theorem 4.2.11d) $= \frac{L}{M}$
Recall:
|x| − |y| $\leq$ ||x| − |y|| $\leq$ |x − y|
|y| $\geq$ |x| − |x − y|
So,
$|g(s_n)| \geq |M| - |M - g(s_n)|$
and since,
$\lim_{n \to \infty} g(s_n) = M \neq 0$
$|g(s_n) - M| < \frac{|M|}{2}$
$-|g(s_n) - M| > \frac{-|M|}{2}$
for n $\geq$ N
So,
$|g(s_n)| > |M| - \frac{|M|}{2} = \frac{|M|}{2} > 0$ for n $\geq$ N □

Also, for the homework:
$\lim_{x \to c}$ P(x) = P(c) where P is a polynomial.

## Example 5.1.14

Since $\lim_{x \to c} x^1$ = c (Example 5.1.3),
then it follows by induction that $\lim_{x \to c} x^n = c^n$ $\forall$ n $\in \mathbb{N}$
**Assume:** for k $\in \mathbb{N}$ , $\lim_{x \to c} x^k = c^k$
**Want to show:** $\lim_{x \to c} x^{k+1} = c^{k+1}$
Define: f(x) = $x^k$, g(x) = x

Then $\lim_{x\to c}$ f(x) = $c^k$, $\lim_{x\to c}$ g(x) = c
So,

$$\lim_{x\to c} x^{k+1} = \lim_{x\to c}(x^k x^1) = \lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = c^k c = c^{k+1}$$

Hence, $\lim_{x\to c}$ $x^n$ = $c^n$, $\forall$ n $\in \mathbb{N}$ by induction.
Combining the result with Theorem 5.1.13, we see that if P is a polynomial and c $\in \mathbb{R}$ , then $\lim_{x\to c}$ P(x) = P(c).
To see this,
**Let:** P(x) = $a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ where $a_i \in \mathbb{R}$ for i = 0, 1, 2, ... n

### Example 5.1.5

Find $\lim_{x\to 1} \frac{2x^2-3x+1}{x-1}$
We need the 0 < part of the limit definition so that the limit can exist even if the function is undefined at that point.
Notice that for x $\neq$ 1,

$$\frac{2x^2 - 3x + 1}{x - 1} = \frac{(x-1)(2x-1)}{(x-1)}$$

So,

$$\lim_{x\to 1} \frac{2x^2 - 3x + 1}{x - 1} = \lim_{x\to 1} 2x - 1 = 2 - 1 = 1$$

If we let q = f(x) = $\frac{2x^2-3x+1}{x-1}$, then f(x) = 2x − 1 with a hole at x = 1.

### One Sided Limits

**Let:** f : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
Then,

i) We write $\lim_{x\to c-}$ f(x) = L iff, for $\epsilon > 0$, $\exists\ \delta > 0$ st |f(x) − L| < $\epsilon$ whenever c − $\delta$ < x < c and x $\in$ D

ii) We write $\lim_{x\to c+}$ f(x) = L iff, for $\epsilon > 0$, $\exists\ \delta > 0$ st |f(x) − L| < $\epsilon$ whenever c < x < c + $\delta$ and x $\in$ D

Of course, if $\lim_{x\to c}$ f(x) = L iff both $\lim_{x\to c-}$ f(x) = $\lim_{x\to c+}$ f(x) = L

## 5.2 Continuous Functions

### Definition 5.2.1

**Let:** f : D $\longrightarrow \mathbb{R}$ and c $\in$ D (we don't know that c is an accumulation point)
We say that f is **continuous at c** if
for any $\epsilon > 0$, $\exists\ \delta > 0$ st

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta \text{ and } x \in D$$

**N.B.**: f(c) must be defined in order for f(x) to be continuous at x = c