HW 9: page 203 - 205, #1, 2, 3(a)(c)(e)(g), 7(c), 13, 16, 18, 19

Chapter 5 Continued:

Theorem 5.1.8

Let f : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
Then,
$\lim_{x\to c}$ f(x) = L $\in \mathbb{R}$ iff for **every** sequence $\{s_n\}$ in D st $s_n \neq c\ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = c$
it follows that $\lim_{n\to\infty} \{f(s_n)\} = L$
So,
$\lim_{x\to c}$ f(x) = L
for $\epsilon > 0$, $\exists\ \delta > 0$ st
$|f(x) - L| < \epsilon$ (i.e. $L - \epsilon < f(x) < L + \epsilon$) whenever $0 < |x - c| < \delta$

Corollary 5.1.9

If f : D $\longrightarrow \mathbb{R}$ and if c $\in$ D$'$,
then
if $\lim_{x\to c}$ f(x) = L, then L is unique.

Proof.
Assume that

$$\lim_{x\to c} f(x) = L_1 \quad (1)$$

and

$$\lim_{x\to c} f(x) = L_2 \quad (2)$$

Let $\{s_n\}$ be a sequence in D st
$s_n \neq c\ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = c$
By **(1)** and Theorem 5.1.8, $\lim_{n\to\infty} f(s_n) = L_1$.
And by **(2)** and Theorem 5.1.8, $\lim_{n\to\infty} f(s_n) = L_2$
However, by Theorem 4.1.14, if a sequence converges, then its limit is unique.
So, $L_1 = L_2$, hence, uniqueness. □

Theorem 5.1.10

Let f : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
Then the following are equivalent:

a. f does not have a limit at c

b. $\exists$ a sequence $\{s_n\}$ in D st $s_n \neq c\ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = c$ but $\{f(s_n)\}$ is not convergent in $\mathbb{R}$

   (looks like the second part of Thm 5.1.8 except the opposite)

Proof.
$\longrightarrow$
We first prove that a $\Rightarrow$ b by using the contrapositive. (i.e. not b implies not a)
Assume **(b)** is false.

Thus, for every sequence $\{s_n\}$ in D st $s_n \neq c \ \forall \ n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = c$
it follows that $\{f(s_n)\}$ converges in $\mathbb{R}$
Want to show: $\lim_{x\to c} f(x)$ exists
Let $\{s_n\}$ and $\{t_n\}$ be sequences in D st $s_n \neq c$ and $t_n \neq c \ \forall \ n \in \mathbb{N}$ in $\lim_{n\to\infty} s_n = c$, $\lim_{n\to\infty} t_n = c$.
Thus,
$\exists \ L_1, L_2 \in \mathbb{R}$ st $\lim_{n\to\infty} f(s_n) = L_1$ and $\lim_{n\to\infty} f(t_n) = L_2$
Want to show: $L_1 = L_2$
Define the sequence $\{u_n\}$ in D by
$\{u_n\} = s_1, t_1, s_2, t_2, ...$
Then $u_n \neq c \ \forall \ n \in \mathbb{N}$ (should be obvious) and $\lim_{n\to\infty} u_n = c$
So $\exists \ L \in \mathbb{R}$ st $\lim_{n\to\infty} f(u_n) = L$
Since s_n and t_n are subsequences of u_n, s_n and t_n must also converge to L.
Thus,
$L_1 = L_2$

———————————Side Note———————————

To see that $\lim_{n\to\infty} u_n = c$,
Let: $\epsilon > 0$
Then $\exists \ N_1, N_2 \in \mathbb{N}$ st $|s_n - c| < \epsilon$ for $n \geq N_1$, and
$|t_n - c| < \epsilon$ for $n \geq N_2$
Let $N = \max \{N_1, N_2\}$

Also, consider $|u_n - c|$
Case:

i) n is even

Then $n = 2k$ for some $k \in \mathbb{N}$ and

$$|u_n - c| = |u_{2k} - c| = |t_k - c| < \epsilon \text{ for } k \geq N$$

So,

$$|u_n - c| < \epsilon \text{ for } n \geq 2N \tag{1}$$

ii) n is odd

Then $n = 2k - 1$ for some $k \in \mathbb{N}$ and

$$|u_n - c| = |u_{2k-1} - c| = |s_k - c| < \epsilon \text{ for } k \geq N$$

So,

$$|u_n - c| < \epsilon \text{ for } n = 2k - 1 \geq 2N - 1 \tag{2}$$

From **(1)** and **(2)**, $\lim_{n\to\infty} u_n = c$

Since $\{f(u_n)\}$ converges to L and $\{f(s_n)\}$, $\{f(t_n)\}$ are subsequences of $\{f(f_n)\}$,
it follows by Theorem 4.4.4 that $L_1 = L_2 = L$
Hence, by Theorem 5.1.8, $\lim_{x\to c} f(x) = L$ □

$\longleftarrow$
Direct proof of **(b)** implies **(a)**.
Assume **(a)** is false.

Then,
$\exists$ L $\in \mathbb{R}$ st $\lim\limits_{x \to c}$ f(x) = L. The result follows directly from Theorem 5.1.8
Recall: a iff b $\longrightarrow$ not a iff not b

Example 5.1.11

Let: $f(x) = \sin(\frac{1}{x})$ for x > 0
Prove that $\lim\limits_{x \to 0}$ f(x) does not exist.

Proof.

Let: $x_n = \frac{2}{n\pi}$ for n $\in \mathbb{N}$
Then,
$\{x_n\}$ is a sequence in D (x > 0) st
$x_n \neq 0\ \forall$ n $\in \mathbb{N}$ and $\lim\limits_{n \to \infty} x_n = 0$, but, $\forall$ n $\in \mathbb{N}$,

$$f(x_n) = \sin(\frac{1}{x_n}) = \sin(\frac{n\pi}{2})$$

Now, $\{f(x_n)\} = 1, 0, -1, 0, 1, 0, -1, 0$ ...
Notice that $\{f(x_n)\}$ does not converge since it possesses subsequences that converge to different limits.
(i.e. $\lim\limits_{k \to \infty} f(x_{2k}) = 0$, $\lim\limits_{k \to \infty} f(x_{4k-3}) = 1$, etc.)
By Theorem 5.1.10, f(x) does not have a limit at x = 0. ☐

Definition 5.1.12

Let f : D $\longrightarrow \mathbb{R}$ and g : D $\longrightarrow \mathbb{R}$
Define:

a. The **sum** f + g : D $\longrightarrow \mathbb{R}$ by (f + g)(x) = f(x) + g(x) $\forall$ x $\in$ D

b. The **product** fg : D $\longrightarrow \mathbb{R}$ by (fg)(x) = f(x)g(x) $\forall$ x $\in$ D

c. The **multiple** kf : D $\longrightarrow \mathbb{R}$ (kf)(x) = kf(x) $\forall$ x $\in$ D, k $\in \mathbb{R}$

d. The **quotient** $\frac{f}{g}$: D $\longrightarrow \mathbb{R}$ $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ $\forall$ x $\in$ D provided that $g(x) \neq 0\ \forall$ x $\in$ D

Theorem 5.1.13

Let f : D $\longrightarrow \mathbb{R}$, g : D $\longrightarrow \mathbb{R}$ and let c $\in$ D$'$
If $\lim\limits_{x\to c}$ f(x) = L and $\lim\limits_{x\to c}$ g(x) = M, then

a. $\lim\limits_{x\to c}$ (f + g) = L + M

b. Let k $\in \mathbb{R}$, $\lim\limits_{x\to c}$ kf = kL

c. $\lim\limits_{x\to c}$ (fg) = LM

d. $\lim\limits_{x\to c}$ $(\frac{f}{g}) = \frac{L}{M}$, provided that M $\neq$ 0

Proof.
(a) through **(c)** are similar to **(d)**.
(d): Let $\{s_n\}$ be a sequence in D st $s_n \neq$ c $\forall$ n $\in \mathbb{N}$ and $\lim\limits_{n\to\infty} s_n$ = c.
Then, by Theorem 5.1.8, $\lim\limits_{n\to\infty}$ f(s_n) = L.
Now, $\lim\limits_{n\to\infty}$ g(x) = M $\neq$ 0.
So $\exists$ N $\in \mathbb{N}$ st

$$g(s_n) \neq 0 \text{ for } n \geq N$$

(ask why? next time)
Then, $\lim\limits_{n\to\infty} (\frac{f}{g})(s_n) = \lim\limits_{n\to\infty} \frac{f(s_n)}{g(s_n)} = \frac{\lim\limits_{n\to\infty} f(s_n)}{\lim\limits_{n\to\infty} g(s_n)}$ (by Theorem 4.2.11d) $= \frac{L}{M}$
Recall:
|x| − |y| $\leq$ ||x| − |y|| $\leq$ |x − y|
|y| $\geq$ |x| − |x − y|
So,
$|g(s_n)| \geq |M| - |M - g(s_n)|$
and since,
$\lim\limits_{n\to\infty} g(s_n) = M \neq 0$
$|g(s_n) - M| < \frac{|M|}{2}$
$-|g(s_n) - M| > \frac{-|M|}{2}$
for n $\geq$ N
So,
$|g(s_n)| > |M| - \frac{|M|}{2} = \frac{|M|}{2}$ for n $\geq$ N □

Also, for the homework:
$\lim\limits_{x\to c}$ P(x) = P(c) where P is a polynomial.