

HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17

For 2(c), see Theorem 1 and Example 9 below

Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

## Section 4.3 Continued

### Theorem 4.4.4

If a sequence  $s_n$  converges to  $s \in \mathbb{R}$ , then every subsequence of  $\{s_n\}$  converges to  $s$  as well.

*Proof.*

Assume  $\{s_n\}$  converges to  $s$ .

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$  st  $|s_n - s| < \epsilon$  for  $n \geq N$  (1)

**Let:**  $\{s_{n_k}\}_{k=1}^{\infty}$  be a subsequence of  $\{s_n\}$

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Side Note

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$\forall k \in \mathbb{N}$ ,

if  $n_k \geq k$  and  $k$  diverges to  $\infty$ , then  $n_k$  diverges to  $\infty$

So, for  $N \in \mathbb{N}$ ,  $\exists k \in \mathbb{N}$  st  $n_k > N$  for  $k \geq K$

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By practice 4.4.3,

$$\lim_{k \rightarrow \infty} n_k = \infty$$

Thus,

$\exists K \in \mathbb{N}$  st  $n_k > N$  for  $k \geq K$  (2)

From (1) and (2),

$$|s_{n_k} - s| < \epsilon \text{ for } k \geq K$$

Hence,

$$\lim_{k \rightarrow \infty} s_{n_k} = s$$

□

**Example 4.4.5** (see page 170, Ex 7(f) for a similar example) (can use this for hw)

Prove that if  $0 < x < 1$ , then  $\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1$

*Proof.*

**Let:**  $x \in \mathbb{R}$ ,  $0 < x < 1$

Define  $s_n = x^{\frac{1}{n}}$  for  $n \in \mathbb{N}$

We shall prove that  $\{s_n\}$  is an increasing sequence that is bounded above.

Notice that for  $n \in \mathbb{N}$ ,

$$\begin{aligned} x^{\frac{1}{n+1}} - x^{\frac{1}{n}} &= x^{\frac{1}{n+1}} \left(1 - x^{\frac{1}{n} - \frac{1}{n+1}}\right) \\ &= x^{\frac{1}{n+1}} (1 - x^{n(n+1)}) \\ &= \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{n+1-n}{n(n+1)} \\ &= \frac{1}{n(n+1)} > 0 \end{aligned}$$

So,  $s_{n+1} \geq s_n, \forall n \in \mathbb{N}$ , i.e.  $\{s_n\}$  is increasing and  $s_n = x^{\frac{1}{n}} < 1 \forall n \in \mathbb{N}$ .

By the Monotone Convergence Theorem (4.3.3?),

$\exists s \in \mathbb{R}$  st

$$\lim_{n \rightarrow \infty} s_n = s$$

Now  $\{s_{2k}\}$  is a subsequence of  $s_n$ .

By Theorem 4.4.4,

$$\lim_{k \rightarrow \infty} s_{2k} = s$$

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Side Note

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This is  $\{s_{n_k}\}$  where  $n_k = 2k \forall k \in \mathbb{N}$

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However,

$$s_{2k} = x^{\frac{1}{2k}} = \left(x^{\frac{1}{k}}\right)^{\frac{1}{2}} = \sqrt{s_k}$$

By Exercise 4.2.6,

$$\lim_{k \rightarrow \infty} s_{2k} = \sqrt{s}$$

Thus,  $s = \sqrt{s}$

But, if a sequence converges, then the limit is unique.

So,

$$s^2 = s$$

$$s(s-1) = 0$$

$$s = 0, 1$$

However, we know that one of those must be wrong.

Since  $s_1 = x^{\frac{1}{1}} = x > 0$  and  $s_n \geq s \forall n \in \mathbb{N}$ ,

we see that  $s \neq 0$ .

Hence,  $s = 1$

□

### Exercise 4.4.6

If  $s_n = (-1)^n \forall n \in \mathbb{N}$ , prove that  $\{s_n\}$  diverges.

Notice that

$$s_{2k} = 1 \forall k \in \mathbb{N}$$

while

$$s_{2k-1} = -1, \forall k \in \mathbb{N}$$

Thus, we have subsequences  $\{s_{2k}\}, \{s_{2k-1}\}$  st

$$\lim_{k \rightarrow \infty} s_{2k} = 1 \text{ and } \lim_{k \rightarrow \infty} s_{2k-1} = -1$$

Hence,  $\{s_n\}$  diverges.

### Theorem 4.4.7

Every bounded sequence has a convergent subsequence.

*Proof.*

**Let:**  $\{s_n\}$  be a bounded sequence

Denote  $S$  as the range of  $\{s_n\}$ :  $S = \{s_n : n \in \mathbb{N}\}$

i)  $S$  is finite.

$$\exists k \in \mathbb{N} \text{ st } S = \{s_1, s_2, \dots, s_k\}$$

Then there is at least one element  $s \in S$  st  $s$  is equal to an infinite number of terms of  $\{s_n\}$ . (i.e. if the range has a finite number of elements, then that means  $s_n$  jumps between each of those elements an infinite number of times. Think of  $1, -1, 1, -1, \dots$ )

Thus,

Choose  $n_1$  such that  $s_{n_1} = s$ .

Then,

Choose  $n_2 > n_1$  such that  $s_{n_2} = s$ .

Inductively,  $\exists s_{n_k} \in S$  such that  $s_{n_k} = s$  and  $n_1 < n_2 < \dots < n_k$

Hence,  $\lim_{k \rightarrow \infty} s_{n_k} = s$

ii)  $S$  is infinite.

Since  $\{s_n\}$  is bounded,  $S$  (our set described above) is also bounded.

By the Bolzano-Weierstrass Theorem,  $\exists s \in S'$  (i.e. an accumulation or limit point:  $s$ )

By HW Exercise 15, page 142 (section 3.4), if  $x \in S'$ , then  $N(x, \epsilon)$  contains an infinite number of points in  $S$ .

Thus,  $\exists s_{n_1} \in S$  st  $s_{n_1} \in N(s, 1)$  (i.e.  $(s - 1, s + 1)$ )

$$|s_{n_1} - s| < \frac{1}{1}$$

Then,

$$\exists n_2 > n_1 \text{ st}$$

$$|s_{n_2} - s| < \frac{1}{2}$$

So, inductively, we can keep doing this (i.e. for  $N(s, \frac{1}{3})$ ,  $N(s, \frac{1}{4})$ , etc)

Thus,

$$|s_{n_k} - s| < \frac{1}{k} \text{ and } n_1 < n_2 < \dots < n_k$$

Hence,  $\lim_{k \rightarrow \infty} s_{n_k} = s$

□

**Theorem 4.4.8**

Every unbounded sequence contains a monotonic sequence that diverges to  $\infty$  or  $-\infty$

*Proof.*

**Let:**  $\{s_n\}$  be a sequence that is unbounded above

Then, for  $m \in \mathbb{R}$ ,  $\exists N \in \mathbb{N}$  st

$s_n > m$  if  $n \geq N$

Notice that this implies that there are an infinite number of terms of  $s_n$  that are strictly larger than  $m$ .

(If there were only a finite number of terms greater than  $m$ , then  $s_n$  wouldn't be unbounded above. There would be a largest term, which would make it have an upper bound.)

Thus,

$\exists n_1 \in \mathbb{N}$  st

$s_{n_1} > 1$

Then,

$\exists n_2 > n_1$  st

$s_{n_2} > 2$

So, inductively, for  $k \in \mathbb{N}$ ,  $\exists n_k \in \mathbb{N}$  st

$$s_{n_k} > k \text{ where } n_1 < n_2 < \dots < n_k$$

Hence, for  $m \in \mathbb{R}$ , the AP guarantees  $k \in \mathbb{N}$  st  $s_{n_k} > k > m$ ,  $\forall k \geq K$ .

This implies that  $\lim_{k \rightarrow \infty} s_{n_k} = \infty$

If  $S$  is bounded above, then  $S$  must be unbounded below, a similar method shows that there is a subsequence

$\{s_{n_l}\}$  st  $\lim_{l \rightarrow \infty} s_{n_l} = -\infty$

□