Theorem 4.3.8

a. If $\{s_n\}$ is an unbounded increasing sequence, then $\lim_{n\to\infty} s_n = \infty$

b. If $\{s_n\}$ is an unbounded decreasing sequence, then $\lim_{n\to\infty} s_n = -\infty$

Proof.

(a)
Since $s_1 \leq s_n \ \forall\ n \in \mathbb{N}$
Thus, if $\{s_n\}$ is unbounded, then it must be unbounded above.
Thus, for $m \in \mathbb{R}$, $\exists\ N \in \mathbb{N}$ st $s_N > m$
Because it's increasing,
$s_n \geq s_N > m$ for $n \geq N$
This is the definition of
Hence,
$\lim_{n\to\infty} s_n = \infty$

(b) is similar.

□

Cauchy Sequences

Definition 4.3.9

A sequence $\{s_n\}$ is **Cauchy** if $\forall\ \epsilon > 0$, $\exists\ N(\epsilon\ ) \in \mathbb{N}$ st

$$|s_n - s_m| < \epsilon, \text{ for m, n } \geq N$$

Lemma 4.3.10

Every convergent sequence is Cauchy.

Proof.

Let: $\{s_n\}$ converge to s.
$\forall\ \epsilon > 0$, $\exists\ N(\epsilon\ ) \in \mathbb{N}$ st

$$|s_n - s| < \frac{\epsilon}{2}, \text{ for n } \geq N$$

Then

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for n, m } \geq N$$

Hence, $\{s_n\}$ is Cauchy. □

Lemma 4.3.11

Every Cauchy sequence is bounded (similar to exam question: Every convergent sequence is bounded)

Proof.

This appeared in a similar context in the HW: Example 13, page 186

□

Theorem 4.3.12 - Cauchy Convergence Criterion

A sequence of real numbers is convergent iff it is a Cauchy sequence.

Proof.

$\longrightarrow$
Assume that $\{s_n\}$ is convergent.
Then, by Lemma 4.3.10, $\{s_n\}$ is Cauchy.
$\longleftarrow$
Conversely, assume that $\{s_n\}$ is Cauchy.
Want to show: $\{s_n\}$ converges
Let: $S = \{s_n : n \in \mathbb{N}\}$ be the range of $\{s_n\}$

i) S is finite.

Thus, $\exists$ k $\in \mathbb{N}$ and $\{n_1, n_2, ... n_k\} \subset \mathbb{N}$ st

$$S = \{s_{n_1}, s_{n_2}, ...s_{n_k}\}$$

Define m:

$$m = \{|s_{n_i} - s_{n_j}| : 1 \leq i \leq j \leq k\}$$
$$m = \{|s_{n_i} - s_{n_j}| : i, j, \in \{n, k\} \text{ and } i \neq j\}$$

Now, for $\epsilon = \frac{m}{2}$, $\exists$ $N(\epsilon) \in \mathbb{N}$ st

$$|s_n - s_m| < \frac{m}{2}, \text{ for n, m } \geq N$$

In particular,

$$|s_n - s_N| < \frac{m}{2} \text{ for } n \geq N \tag{1}$$

Now, $\exists$ l $\in \{1, 2, ... k\}$ st $s_N = s_{n_l}$

Thus, **(1)** implies that

$$|s_n - s_{n_l}| < \frac{m}{2} \text{ for } n \geq N$$

Thus, $s_n = s_{n_l}$ $\forall$ n $\geq$ N

Hence, $\lim\limits_{n\to\infty} s_n = s_{n_l}$

ii) S is infinite.

Since $\{s_n\}$ is Cauchy, it follows by Lemma 4.3.11 that S is bounded.

By the Bolzano-Weierstrass theorem,

$\exists$ s mem $\mathbb{R}$ st

s $\in$ S$'$ (i.e. s is an accumulation point of S)

Want to show: $\lim_{n\to\infty} = s$

For $\epsilon > 0$, $\exists$ N(ϵ) $\in \mathbb{N}$ st

$$|s_n - s_m| < \frac{\epsilon}{2} \text{ for n, m} \geq N \tag{1}$$

Also by Exercise 15, Section 1.4, page 142, since s $\in$ S$'$,

Every deleted neighborhood of s, N*(s,ϵ) contains **an infinite number** of points from S.

Since there are an infinite number of points in N*(s,ϵ), it's totally reasonable that there are an infinite number of points in N*(s,$\frac{\epsilon}{2}$)

Thus, $\exists$ m $\in \mathbb{N}$ with M $\geq$ N st

$s_m \in$ N(s,$\frac{\epsilon}{2}$)

So,

$$|s - s_m| < \frac{\epsilon}{2} \tag{2}$$

From **(1)** and **(2)**,

$$\begin{aligned}|s - s_n| &= |(s - s_m) + (s_m - s_n)| \\ &\leq |s - s_m| + |s_m - s_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ &\forall n \geq N\end{aligned}$$

Hence, $\lim_{n\to\infty} s_n = s$, which completes the proof.

□

————Side Note————

Better ratio test:
$\{s_n\}$ is a sequence.
Test:
$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = L < 1$
If $\lim_{n\to\infty} |s_n| = 0$,
$\lim_{n\to\infty} |s_n| = 0 = \lim_{n\to\infty} |s_n - 0| = \lim_{n\to\infty} s_n = 0$
The reason is because, if you're not careful, you can conclude something like, say, $\lim_{n\to\infty} s_n = (-2)^n = 0$
$s_n = (-2)^n$
$\frac{s_{n+1}}{s_n} = \frac{(-2)^{n+1}}{(-2)^n} = -2 < 1$
which would tell you, in theory, that the limit is 0. Which is **not** true.

Example 4.3.13

Show that the harmonic series $s_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

Solution:
Let n $\in \mathbb{N}$ and

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{m}$$

Then, for m > n,

$$\begin{aligned} |s_n - s_m| &= s_m - s_n \\ &= \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+(m-n)} \\ &> \frac{m-n}{n+(m-n)} = \frac{m-n}{m} \end{aligned}$$

So, if m = 2n, then

$$|s_n - s_{2n}| > \frac{2n-n}{2n} = \frac{n}{2n} = \frac{1}{2}, \forall n \in \mathbb{N}$$

This tells me that $\{s_n\}$ is **not** Cauchy.
Hence, by the Cauchy Convergence Criterion, $|s_n|$ diverges.
Notice that $\{s_n\}$ is a monotonically increasing sequence that is unbounded.
So, by Theorem 4.3.8(a), $\lim_{n\to\infty} s_n = \infty$

4.4.1 Subsequences

Definition 4.4.1

Let: $\{s_n\}_{n=1}^{\infty}$ be a sequence
Also, let $\{n_k\}$ be a sequence $\in \mathbb{N}$ st

$$n_1 < n_2 < n_3 ...$$

The sequence $\{s_{n_k}\}_{k=1}^{\infty}$ is called a **subsequence** of $\{s_n\}$.
Notice that, in this case, $n_k \geq k$ (i.e. $k \leq t_k$) $\forall$ k $\in \mathbb{N}$
Thus, $\lim_{n\to\infty} n_k = \infty$

———————————Side Note———————————

If $s_n \leq t_n$ $\forall$ n $\in \mathbb{N}$, and
if $\lim_{n\to\infty} s_n = \infty$,
then $\lim_{n\to\infty} t_n = \infty$

Practice 4.4.3

Let $\{n_k\}$ be a sequence in $\mathbb{N}$ such that $n_k < n_{k+1}$ $\forall$ k $\in \mathbb{N}$.
Use induction to prove that $n_k \geq k$, $\forall$ k $\in \mathbb{N}$
Solution:
Notice that $1 \leq n_1$
For $l \in \mathbb{N}$,
assume that $l \leq n_l$
Now, consider $l + 1 \leq n_l + 1$

Side Note

$n_k < n_{k+1}$
$n_k + 1 \leq n_{k+1}$

So, $l + 1 \leq n_l + 1 \leq n_{l+1}$
Hence,
$k \leq n_k$ $\forall$ $k \in \mathbb{N}$