Theorem 3.2.8 - pg 118

Let x, y $\in \mathbb{R}$

a. If x $\leq$ y + ϵ $\forall$ $\epsilon > 0$, then x $\leq$ y

b. If $|x - y| \leq \epsilon$ $\forall$ $\epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

a)

If x $\leq$ y + ϵ $\forall$ $\epsilon > 0$, then x $\leq$ y

Proof.

Suppose that:

$$x > y$$

$$x - y > 0$$

Let

$$\epsilon = \frac{x+y}{2} > 0$$

See that

$$\begin{aligned} y + \epsilon & \\ &= y + \frac{x+y}{2} \\ &= y + \frac{x}{2} - \frac{y}{2} \\ &= \frac{x}{2} + \frac{y}{2} \\ &< \frac{x}{2} + \frac{x}{2} \\ &< x \\ y + \epsilon < x & \quad \textbf{(1)} \end{aligned}$$

Thus, by contrapositive, the result is true.

□

b)

If $|x - y| \leq \epsilon \; \forall \; \epsilon > 0$, then $|x - y| = 0$ or, evidently, $x = y$

Proof.

Suppose that:

$$|x - y| > 0$$

Let

$$\epsilon = \frac{|x - y|}{2}$$

See that

$$\begin{aligned}
&1 > \frac{1}{2} \\
&|x - y| > \frac{1}{2}|x - y| \\
&|x - y| > \epsilon
\end{aligned}$$

Thus, by contrapositive, the result is true.

□

Definition 3.2.9

If x ∈ ℝ,

$$|x| = \begin{cases} x, & \text{if } x \geq 0. \\ -x, & \text{if } x < 0. \end{cases}$$

Theorem 3.2.10

Let x, y ∈ ℝ and a ≥ 0

Then

a. $|x| \geq 0$

b. $|x| \leq a$ iff $-a \leq x \leq a$

c. $|xy| = |x||y|$

d. $|x + y| \leq |x| + |y|$ (equality holds only if signs are the same)

a)

$|x| \geq 0$

Proof.

Case:

i) $x \geq 0$:
then $|x| = x \geq 0$

ii) $x < 0 \Rightarrow -x > 0$
then $|x| = -x \geq 0$

Hence, result □

b)

$|x| \leq a$ iff $-a \leq x \leq a$

Since it's a biconditional, first we prove p ⇒ q, then q ⇒ p.

Proof.

Notice that:

$$-a \leq -|x|$$

Case:

i) $x \geq 0$
then $0 \leq x = |x|$
and $\therefore |x| \leq a$

Also, since $x = |x| \geq 0$, $-a \leq x$ or $-a \leq 0$ $-a \leq x \leq a$

ii) $x < 0$ $|x| = -x \leq ax \geq -a \therefore -a \leq x - a \leq x \leq a$

Hence, result.
$\longleftarrow$
Conversely, we shall prove that q $\Rightarrow$ p
Suppose: $-a \leq x \leq a$
Then:

i) x ≥ 0, then $|x| = x \leq a$

ii) x < 0, then $|x| = -x \leq a$

Hence, result.

c)

$|xy| = |x||y|$
Notice that if x $= 0$ (p) or y $= 0$ (q), then $|xy| = 0 = |x||y|$.
WLOG, assume that not [p or q] = not p $\cap$ not q.

i) $x > 0$ and $y > 0$ then $|x| = x$, $|y| = y$ Also, $xy > 0$ So, $|xy| = xy = |x||y|$

ii) $x < 0$, $y < 0$ then $|x| = -x, |y| = -y, xy > 0$ So, $|xy| = xy = -|x|(-|y|) = |x||y|$

iii) $x > 0$, $y < 0$ OR $y > 0$, $x < 0$ WLOG, let $x > 0$, $y < 0$ $|xy| = |x||y|$ $|yx| = |y||x|$ $|x| = x$, $|y| = -y$, $xy < 0$ So, $|xy| = -(xy)$ $-[|x|(-|y|)]$ $-[-|x||y|]$ $|x||y|$

d)

$|x + y| \leq |x| + |y|$
Let: Z $= x + y$, and a $= |x| + |y|$
If $a \geq 0$, then $|Z| \leq a$ iff $-a \leq Z \leq a$
$-(|x| + |y|) \leq x + y \leq |x| + |y|$
From b), since $|x| + |y| \geq 0$,

Want to show: $-(|x| + |y|) \leq x + y \leq |x| + |y|$
Notice: $-|x| \leq x \leq |x|$
$|x| = x$ or $|x| = -x$ or $-|x| = x$

Then
$-|x| - |y| \leq x + y \leq |x| + |y|$
$-(|x| + |y|) \leq x + y \leq |x| + |y|$
By b), this is equivalent to
$|x + y| \leq |x| + |y|$

$\square$