HW 9: page 203 - 205, #1, 2, 3(a)(c)(e)(g), 7(c), 13, 16, 18, 19

Exercise 1

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D$'$
Mark each statement True or False. Justify each answer.

a. $\lim_{x\to c}$ f(x) = L iff $\forall\ \epsilon > 0$, $\exists$ a $\delta > 0$ st |f(x) − L| < ϵ whenever x $\in$ D and |x − c| < δ

False. Let f(c) $\neq$ L as a counter example.

b. $\lim_{x\to c}$ f(x) = L iff for every deleted neighborhood U of c, there exists a neighborhood V of L st f(U $\cap$ D) $\subset$ V

True, by Theorem 5.1.2 (since it's iff).

c. $\lim_{x\to c}$ f(x) = L iff for every sequence $\{s_n\}$ in D that converges to c with $s_n \neq$ c $\forall$ n, the sequence $\{f(s_n)\}$ converges to L.

True, by Theorem 5.1.8.

d. If f does not have a limit at c, then $\exists$ a sequence $\{s_n\}$ in D with each $s_n \neq$ c st $\{s_n\}$ converges to c, but $\{f(s_n)\}$ is divergent.

True by Theorem 5.1.10(b).

Exercise 2

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D$'$
Mark each statement True or False. Justify each answer.

a. For any polynomial P and any c $\in \mathbb{R}$, $\lim_{x\to c}$ P(x) = P(c)

True.

$\lim_{x\to c}$ P(x) = P(c) iff $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ st x $\in$ D and 0 < |x − c| < δ $\Rightarrow$ |P(x) − P(c)| < ϵ

From lecture: $\lim_{x\to c}$ P(x) = P(c) where P is a polynomial.

b. For any polynomials P and Q, and any c $\in \mathbb{R}$,

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

False. This is not the case if Q(c) is 0.

c. In evaluating $\lim_{x\to a-}$ f(x) we only consider points that are greater than a.

False, by the definition of One Sided Limits.

d. If f is defined in a deleted neighborhood of c, then $\lim_{x\to c}$ f(x) = L iff $\lim_{x\to c+}$ f(x) = $\lim_{x\to c-}$ f(x) = L

True. If f is defined in a deleted neighborhood of c, then by Theorem 5.1.2,

$\lim_{x\to c}$ f(x) = L iff $\lim_{x\to c+}$ f(x) = $\lim_{x\to c-}$ f(x) = L

Exercise 3(a)(c)(e)(g)

Determine the following limits:

a. $\lim_{x\to 1} \frac{3x^2+5}{x^3+1}$

$$\lim_{x\to 1} \frac{3x^2+5}{x^3+1} = \frac{3(1)^2+5}{1^3+1} = \frac{8}{2} = 4$$

b. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{1+1} = \frac{1}{2}$$

c. $\lim_{x\to 0} \frac{x^2+5x}{x^2-2}$

$$\lim_{x\to 0} \frac{x^2+5x}{x^2-2} = \frac{0^2+5(0)}{0^2-2} = \frac{0}{-2} = 0$$

d. $\lim_{x\to 0-} \frac{4x}{|x|}$

Since we're only taking x $<$ 0, $\frac{4x}{|x|}$ will always be -4, so the limit from the left side is -4.

Exercise 7(c)

Find the following limit and prove your answer.
$\lim_{x\to c} \sqrt{x}$, where c $\geq$ 0
Let: f(x) = $\sqrt{x}$
I think the limit is just $\sqrt{c}$.
$\lim_{x\to c} \sqrt{x} = \sqrt{c}$ iff
$\forall\ \epsilon > 0, \exists\ \delta > 0$ st x $\in$ D and $0 <$ |x − c| $< \delta \ \Rightarrow |\sqrt{x} - \sqrt{c}| < \epsilon$
We know that since the domain for $\sqrt{x}$ is x $\geq$ 0, D is {x : x $\geq$ 0}
Let: $\delta = \epsilon^2$
Then,
$\forall\ \epsilon > 0, \exists\ \delta > 0$ st x $\in$ D and $0 <$ |x − c| $< |\sqrt{x} - \sqrt{c}||\sqrt{x} - \sqrt{c}| \ \Rightarrow |\sqrt{x} - \sqrt{c}| < \epsilon$
$\forall\ \epsilon > 0, \exists\ \delta > 0$ st x $\in$ D and $0 <$ |x − c| $< |x^2 - 2\sqrt{x}\sqrt{c} + c^2| \ \Rightarrow |\sqrt{x} - \sqrt{c}| < \epsilon$
Hence, result.*

Exercise 13

Let f, g, and h be functions from D into $\mathbb{R}$, and let $c \in D'$
Assume: $f(x) \leq g(x) \leq h(x) \ \forall \ x \in D$ with $x \neq c$
Assume: $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$
Prove that $\lim_{x \to c} g(x) = L$
We know that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$, so:
$\forall \ \epsilon > 0, \exists \ \delta_f > 0$ st $x \in D$ and $0 < |x - c| < \delta_f \ \Rightarrow |f(x) - L| < \epsilon$
$\forall \ \epsilon > 0, \exists \ \delta_h > 0$ st $x \in D$ and $0 < |x - c| < \delta_h \ \Rightarrow |h(x) - L| < \epsilon$
Let: $\delta = \min \{\delta_h, \delta_f\}$
Now:
$\forall \ \epsilon > 0, \exists \ \delta > 0$ st $x \in D$ and $0 < |x - c| < \delta \ \Rightarrow |f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$
$|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$
$-\epsilon < f(x) - L < \epsilon$ and $-\epsilon < h(x) - L < \epsilon$
$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon$
$-\epsilon < g(x) - L < \epsilon$
$|g(x) - L| < \epsilon$
Hence,
$\lim_{x \to c} g(x) = L$

Exercise 16

Let: $f : D \longrightarrow \mathbb{R}$ and $c \in D'$
Assume: $\lim_{x \to c} f(x) > 0$
Prove that $\exists$ a deleted neighborhood U of c st $f(x) > 0 \ \forall \ x \in (U \cap D)$
Let: $\lim_{x \to c} f(x) = L$, where $L > 0$
So,
$\forall \ \epsilon > 0, \exists \ \delta > 0$ st $x \in D$ and $0 < |x - c| < \delta \ \Rightarrow |f(x) - L| < \epsilon$
Define ϵ st $L - 2\epsilon = 0$
Notice that $f(x) > 0, \forall \ f(x) \in V = N(L, \epsilon)$.
By Theorem 5.1.2, $\exists \ \delta > 0$ st $U = N^*(c, \delta)$ and $f(U \cap D) \subset V$
Hence, result.

Exercise 18

Let: f : D $\longrightarrow \mathbb{R}$ and c $\in$ D′
Assume: f has a limit at c (i.e. $\lim_{x\to c}$ f(x) = L)
Prove that f is bounded on a neighborhood of c.
That is, prove that $\exists$ a neighborhood U of c and a real number M st $|f(x)| \leq M \ \forall \ x \in (U \cap D)$

Since $\lim_{x\to c}$ f(x) = L,
$\forall \ \epsilon > 0, \exists \ \delta > 0$ st x $\in$ D and $0 < |x - c| < \delta \ \Rightarrow |f(x) - L| < \epsilon$
and,
for each neighborhood V of L, $\exists$ a deleted neighborhood U of c st $f(U \cap D) \subset V$.
Let: V be a neighborhood of L, and U be the deleted neighborhood U of c st $f(U \cap D) \subset V$
Pick x $\in U \cap D$
(Can we assume that V is a neighborhood with a finite boundary here since the definition of a neighborhood is with some real number $\epsilon > 0$?)
If we let LB = min V, and UB = max V, then we see that V is bounded.
We also know that since $f(x) \in V \ \forall \ x \in U \cap D$, that means f(x) is bounded.
We see that $\exists$ M $\in \mathbb{R}$ st $|f(x)| \leq M$, and that completes our proof.

Exercise 19

Assume: f : $\mathbb{R} \longrightarrow \mathbb{R}$ is a function st f(x + y) = f(x) + f(y) $\forall$ x, y $\in \mathbb{R}$
Prove that f has a limit at 0 iff f has a limit at every point c $\in \mathbb{R}$

$\longrightarrow$
f has a limit at 0, so,
$\forall \ \epsilon > 0, \exists \ \delta > 0$ st $(x + y) \in D$ and $0 < |x + y| < \delta \ \Rightarrow |f(x + y) - L| < \epsilon$
Notice:
D = $\mathbb{R}$ and f(x + y) − f(y) = f(x)
So,
$\forall \ \epsilon > 0, \exists \ \delta > 0$ st x $\in$ D and $0 < |x| < \delta \ \Rightarrow |f(x) - L| < \epsilon$
By the definition of f, and since x + c $\in$ D,
$0 < |x - c| < \delta \ \Rightarrow |f(x) + f(-c) - L| < \epsilon$
If we let L(c) = L − f(−c), then:
$0 < |x - c| < \delta \ \Rightarrow |f(x) - L(c)| < \epsilon$

$\longleftarrow$
f has a limit at every point c $\in \mathbb{R}$
$0 \in \mathbb{R}$
Hence,
f has a limit at 0