HW 8: pages 193, #1, 2, 3, 5, 9, 10, 17
For 2(c), see Theorem 1 and Example 9 from Lecture 15
Make sure when you do these problems, justify the answer by either writing down the theorem name or providing a counter example.

Exercise 1

Mark each statement True or False. Justify each answer.

a. A sequence (s_n) converges to s iff every subsequence of (s_n) converges to s.

   True. By Theorem 4.4.4.

b. Every bounded sequence is convergent.

   False.

   Counter example: (s_n) = $(-1)^n$

c. Let (s_n) be a bounded sequence. If (s_n) oscillates, then the set S of subsequential limits of (s_n) contains at least two points.

   True. If S oscillates, then lim inf S $<$ lim sup S. This implies that these are two different points.

d. Let (s_n) be a bounded sequence and let m = lim sup s_n.

   Then, $\forall$ $\epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st N $\geq$ n implies $s_n > m - \epsilon$

   True.

   Proof.
   Let: $\epsilon > 0$

   Since s_n is bounded, let S be the set containing the range of s_n.

   By definition, $\exists$ some s_{n_k} st lim s_{n_k} = m where k $\in \mathbb{N}$

   Since lim s_{n_k} = m,

   $\exists$ N $\in \mathbb{N}$ st N $\geq n_k$ implies $|s_{n_k} - m| < \epsilon$

   $|s_{n_k} - m| < \epsilon$

   $-\epsilon < s_{n_k} - m < \epsilon$

   $m - \epsilon < s_{n_k} < m + \epsilon$ **(1)**

   So, by **(1)**,

   $\exists$ some N $\in \mathbb{N}$ st n $\geq$ N implies $s_n > m - \epsilon$

   □

e. If (s_n) is unbounded above, then (s_n) contains a subsequence that has ∞ as a limit.

   True. By Theorem 4.4.8.

Exercise 2

Mark each statement True or False. Justify each answer.

a. Every sequence has a convergent subsequence.

   False. Let $s_n = n$

b. The set of subsequential limits of a bounded sequence is always nonempty.

   True. By Theorem 4.4.8

c. (s_n) converges to s iff $\lim\inf s_n = \lim\sup s_n = s$

   True. By Definition 4.4.9 and exercise 9.

d. Let (s_n) be a bounded sequence and let $m = \lim\sup s_n$. Then, $\forall\ \epsilon > 0$, there are infinitely many terms in the sequence greater than $m - \epsilon$.

   True. By Theorem 4.4.7, s_n has a convergent subsequence.

   Let t_n be a subsequence of s_n st $\lim\limits_{n\to\infty} t_n = m$

   By definition,

   $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - m| < \epsilon$

   so,

   $-\epsilon < t_n - m < \epsilon$

   $m - \epsilon < t_n$

   Pick ϵ_2 to be $\frac{\epsilon}{2}$

   Then,

   $\exists\ N(\epsilon_2)$ st $m - \epsilon < t_{N(\epsilon_2)}$

   Inductively, we can let $\epsilon_3 = \frac{\epsilon_2}{2}$, and so on.

   Hence, since there are infinitely many terms in t_n greater than $m - \epsilon$, the same is true for s_n.

e. If (s_n) is unbounded above, then $\lim\inf s_n = \lim\sup s_n = \infty$

   True.

   Suppose: s_n has a subsequence t_n such that $\lim\limits_{n\to\infty} t_n = t$ where $t \neq \infty$ (but could be negative infinity)

   So,

   $\forall\ \epsilon > 0, \exists\ N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - t| < \epsilon$

   Notice also, that since s_n is unbounded above,

   $\forall\ m \in \mathbb{R}\ , \exists\ N_m \in \mathbb{N}$ st $s_{N_m} > m$

   That means that $\exists$ some N for t_n st $t_N > m$

   If we let $m = t$, then

   $\exists$ some N_1 for t_n st $t_{N_1} > t = m$

   If we let $m = t + 1$, then

   $\exists$ some N_2 for t_n st $t_{N_2} > m = t + 1$

   Inductively, t_n has an infinite amount of values above t, and is increasing: a contradiction.

   Thus, t_n is unbounded above.

Exercise 3

For each sequence, find the set S of subsequential limits, the limit inferior, and the limit superior.

a. $s_n = 1 + (-1)^n$

   $S = \{0, 2\}$, $s_* = 0$, $s^* = 2$

b. $t_n = (0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7})$

   $S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, \frac{1}{6}, \frac{6}{7}\}$, $s_* = 0$, $s^* = \frac{6}{7}$

c. $u_n = n^2(-1 + (-1)^n)$

   $S = \{0\}$, $s_* = -\infty$, $s^* = 0$

d. $v_n = n \sin \frac{n\pi}{2}$

   $S = \{0\}$, $s_* = -\infty$, $s^* = \infty$

Exercise 5

Use exercise 4.3.14 to find the limit of each sequence:
Known: $t_n = (1 + \frac{1}{n})^n$ and $\lim\limits_{n\to\infty} t_n = e$

a. $s_n = (1 + \frac{1}{2n})^{2n}$

   We can just think of s_n as a subsequence of t_n (the original e sequence),

   so therefore it has the same limit: e.

b. $s_n = (1 + \frac{1}{n})^{2n}$

   $= ((1 + \frac{1}{n})^n)^2$

   so, $\lim\limits_{n\to\infty} s_n = e^2$

c. $s_n = (1 + \frac{1}{n})^{n-1}$

   $= (1 + \frac{1}{n})^n(1 + \frac{1}{n})^{-1}$

   so, $\lim\limits_{n\to\infty} s_n = e * 1 = e$

d. $s_n = (\frac{n}{n+1})^n$

   $= \frac{1}{(\frac{n+1}{n})^n}$

   $= \frac{1}{(1+\frac{1}{n})^n}$

   so, $\lim\limits_{n\to\infty} s_n = \frac{1}{e}$

e. $s_n = (1 + \frac{1}{2n})^n$

   $= ((1 + \frac{1}{2n})^{2n})^{\frac{1}{2}}$

   so, $\lim\limits_{n\to\infty} s_n = \sqrt{e}$

f. $s_n = (\frac{n+2}{n+1})^{n+3}$

   $= (\frac{n+2}{n+1})^n(\frac{n+2}{n+1})^3$

   $= (\frac{n}{n+1} + \frac{2}{n+1})^n(\frac{n+2}{n+1})^3$

   Now, $\lim\limits_{n\to\infty} (\frac{n}{n+1} + \frac{2}{n+1})^n(\frac{n+2}{n+1})^3 = (e + 0) \times 1$ by **(d)**

   so, $\lim\limits_{n\to\infty} s_n = e$

Exercise 9

Let (s_n) be a bounded sequence.
Assume: lim inf s_n = lim sup s_n = s
Prove that (s_n) is convergent and that lim s_n = s

Let $S \subset \mathbb{R}$ be the range of limits for any subsequence of s_n.
Since lim inf s_n = s, inf S = s.
Since lim sup s_n = s, sup S = s.
By Corollary 4.4.12, S contains s.
Since inf S = sup S = s, the range of S is just {s}. **(1)**
Since s_n is bounded, it can't diverge to ∞ or $-\infty$.
However, suppose s_n diverges in general.
Then, $\exists\ \epsilon\ (s_n)$ st $|s_n - s| > \epsilon\ (s_n)$ for all $n \geq$ some $N \in \mathbb{N}$
Since there are infinitely many $n \geq N$, $\exists\ s_{n_k}$ (a subsequence of s_n) st
$|s_{n_k} - s| \geq \epsilon\ (s_n)$ where $n_k = N + k$, $k \in \mathbb{N}$
Since s_{n_k} is bounded (because s_n is bounded), it itself has a convergent subsequence (for notation reasons lets call it t_{n_k})
Notice that t_{n_k} is a convergent subsequence of s_n, but it's limit is not s (since $\exists$ an ϵ st $|s_n - s| > \epsilon$), a contradiction.
Hence, s_n must converge to s.

Alternative way:

Using Theorem 4.4.11(a) and (c) (or (a)(i) / (b)(i) according to Welsh):
(a): $\forall\ \epsilon > 0$, $\exists\ N_1\ (\epsilon\ ) \in \mathbb{N}$ st

$$s_n < s^* + \epsilon \text{ for } n \geq N_1(\epsilon)$$

(c): $\forall\ \epsilon > 0$, $\exists\ N_2\ (\epsilon\ ) \in \mathbb{N}$ st

$$s_n > s_* - \epsilon \text{ for } n \geq N_2(\epsilon)$$

Let $N = \max\{N_1, N_2\}$ st
$s - \epsilon < s_n - s < s + \epsilon$, for $n \geq N$
Hence,
$|s_n - s| < \epsilon$, for $n \geq N$.
So,
$\lim\limits_{n\to\infty} s_n = s$

Exercise 10

Assume: $x > 1$
Prove that lim $x^{\frac{1}{n}} = 1$
$\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1$ if
$\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ st $N \geq n$ implies $|x^{\frac{1}{n}} - 1| < \epsilon$
Let: $\epsilon > 0$
$|x^{\frac{1}{n}} - 1| < \epsilon$
Since $x > 1$ and $n \in \mathbb{N}$,
$x^{\frac{1}{n}} - 1 < \epsilon$
$x^{\frac{1}{n}} < \epsilon + 1$
$(x^{\frac{1}{n}})^n < (\epsilon + 1)^n$

x $< (\epsilon + 1)^n$
ln x < n ln $(\epsilon + 1)$
$\frac{\ln x}{\ln (\epsilon + 1)} < n$
So, if $\frac{\ln x}{\ln (\epsilon + 1)} <$ N,
then ∃ N st $|x^{\frac{1}{n}} - 1| < \epsilon$
Hence, result.

Alternative way:

Recall: $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$, $0 < x < 1$
$\lim_{n\to\infty} (\frac{1}{x})^{\frac{1}{n}} = 1$
But,

$$(\frac{1}{x})^{\frac{1}{n}} = \frac{1^{\frac{1}{n}}}{x^{\frac{1}{n}}} = \frac{1}{x^{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{1}{x^{\frac{1}{n}}} = 1$$

$$\lim_{n\to\infty} \frac{1}{\frac{1}{x^{\frac{1}{n}}}} = \frac{1}{1} = 1$$

Hence,

$$\lim_{n\to\infty} x^{\frac{1}{n}} = 1$$

Exercise 17

Prove that if lim sup $s_n = \infty$ and $k > 0$, then lim sup $(ks_n) = \infty$

Side Note

Question: Is it a valid proof to say that since

$$t_n = \sum_{i=1}^{n} \frac{1}{n}$$

is the slowest possible diverging sequence (without constants of course),
since

$$\lim_{n\to\infty} kt_n = k\infty = \infty$$

then $\lim_{n\to\infty}$ of k × any sequence diverging to ∞ is also ∞?
So, therefore lim sup (k * any sequence diverging to ∞) is also ∞ ?

Let: t_n be a subsequence of s_n st $\lim_{n\to\infty} t_n = \infty$
Algebraically,
k $\lim_{n\to\infty} s_n = \lim_{n\to\infty} ks_n = \lim_{n\to\infty} ks_1, ks_2, ks_3... \ ks_n = k\infty = \infty$
Since the limit of any subsequence is the same as the limit of the sequence, and by Theorem 4.4.14,
k $\lim_{n\to\infty} t_n = \lim_{n\to\infty} kt_n = \lim_{n\to\infty} kt_1, kt_2, kt_3... \ kt_n = k\infty = \infty$
So, since kt_n is a subsequence of ks_n,
lim sup $(ks_n) = \infty$