Homework 7: pages 184 - 185 numbers 1, 2(a)(b), 3(e), 4, 10, 13, 14 $\longleftarrow$ 14 is difficult, but not impossible!
(want to show that lim $(1 + \frac{1}{n})^n$ exists)
Hint:
$(1 + \mathrm{b})^n = 1 + \mathrm{nb} + \frac{n(n-1}{2!}\mathrm{b}^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}\mathrm{b}^r + ... + \mathrm{b}^n$
In our problem, b = $\frac{1}{n}$
Look at it as $1 + \sum_{r=1}^{n} \frac{n(n-1)...(n-(r-1))}{r!} \frac{1}{n^r}$
$(1 + \frac{1}{n})^n$ goes in there somewhere somehow.

Problem 1

Mark each statement True or False. Justify each answer.

a. If a monotone sequence is bounded, then it is convergent.

   True

   by Theorem 4.3.3

b. If a bounded sequence is monotone, then it is convergent.

   True

   by Theorem 4.3.3

c. If a convergent sequence is monotone, then it is bounded.

   True

   by Theorem 4.3.3

Problem 2(a)(b)

Mark each statement True or False. Justify each answer.

a. If a convergent sequence is bounded, then it is monotone.

   False.

   Counterexample: s_n =

$$(-1)^n \frac{1}{n}$$

b. If (s_n) is an unbounded increasing sequence, then lim $s_n = +\infty$

   True.

   By Theorem 4.3.8

   Assume: (s_n) is an unbounded, increasing sequence.

   Then, $\forall$ n $\in \mathbb{N}$, $s_n \leq s_{n+1}$

   and

   $\forall$ m $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st. n $\geq$ N implies $s_n >$ m

   By Definition 4.2.9:

   We say a sequence diverges to ∞ if $\forall$ m $\in \mathbb{R}$, $\exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $s_n >$ m

   Hence, result.

Problem 3(e)

Prove that each sequence is monotone and bounded. Then, find the limit.
(e) $s_1 = 5$ and $s_{n+1} = \sqrt{4s_n + 1}$ for $n \geq 1$
s_n is monotone if it's either increasing or decreasing.
$s_1 = 5$, $s_2 = \sqrt{21} = 4.58257569496$, $s_3 = \sqrt{4\sqrt{21}+1} = \sqrt{\sqrt{336}+1} = \sqrt{19.3303028} = 4.39662402304$
Hmm, limit's probably 4. Let's see.

Conjecture
$\{s_n\}$ is decreasing and $4 \leq s_n \leq 5$, $\forall$ $n \in \mathbb{N}$
P(n) (Proposition as a function of n):
$s_n \geq s_{n+1}$, $\forall$ $n \in \mathbb{N}$
$s_1 = 5 > \sqrt{21} = s_2$
Suppose that, $\forall$ $k \in \mathbb{N}$,

$$s_k = \sqrt{4s_{k-1}+1} \geq \sqrt{4s_k + 1} = s_{k+1}$$

Now,

$$s_{k+1} = \sqrt{4s_k+1} \geq \sqrt{4s_{k+1}+1} = s_{k+2}$$

So,

$$s_k \geq s_{k+1}$$

Hence, by induction, P(n): $s_n \geq s_{n+1}$ is true $\forall$ $n \in \mathbb{N}$
Q(n): $s_n \geq 4$ $\forall$ $n \in \mathbb{N}$
$s_1 = 5 > 4$
Assume for $k \in \mathbb{N}$ that $s_k > 4$

$$s_{k+1} = \sqrt{4s_k+1} > \sqrt{4(3.75)+1} = 4$$

Hence, by induction, Q(n): $s_n > 4$ is true $\forall$ $n \in \mathbb{N}$
By the Montone Convergence Theorem,
$\exists$ $s \in \mathbb{R}$ st
$\lim\limits_{n\to\infty} s_n = s$
By HW problem 11, page 170.
Thus,
$\lim\limits_{n\to\infty} s_{n+1} = \lim\limits_{n\to\infty} s_n = s$
So, we claim that $\lim\limits_{n\to\infty} s_{n+1} = s = \lim\limits_{n\to\infty} \sqrt{4s_n+1} = \sqrt{4s+1}$
From Example 4.2.6,
$\lim\limits_{n\to\infty} \sqrt{t_n} = \sqrt{t}$ if $\lim\limits_{n\to\infty} t_n = t$
Also, by Theorem 4.2.1 (b), $\lim\limits_{n\to\infty} \sqrt{1+s_n} = \sqrt{1+s}$
(which is like saying $\lim\limits_{n\to\infty} t_n = t$)
Hence,

$$\begin{aligned} s &= \sqrt{4s+1} \\ s^2 &= 4s+1 \\ s^2 - 4s - 1 &= 0 \\ s &= \frac{4 \pm \sqrt{20}}{2} \end{aligned}$$

But one of those limits can't be true since limits are unique.
Since $s_n \geq 0, \forall$ n $\in \mathbb{N}$,
then $\lim\limits_{n\to\infty} s_n = s \geq 0, \forall$ n $\in \mathbb{N}$
(By Corollary 4.2.5)
Hence,
s = $\frac{4+\sqrt{20}}{2} = 2 + \sqrt{5}$

Problem 4

Find an example of a sequence of real numbers satisfying each set of properties.

a. Cauchy, but not monotone.

   $s_n = (-1)^n \frac{1}{n}$

b. Monotone, but not cauchy.

   s_n = n

c. Bounded, but not cauchy.

   $s_n = (-1)^n$

Problem 10

a. Suppose that |r| < 1. Recall from Exercise 3.1.7 that

$$1 + r + r^2 + ...r^n = \frac{1 - r^{n-1}}{1 - r}$$

Find $\lim\limits_{n\to\infty}$ (1 + r + r^2 + ... r^n).

In other words, find $\lim\limits_{n\to\infty} \frac{1-r^{n-1}}{1-r}$

By Theorem 4.2.1, $\lim\limits_{n\to\infty} \frac{1-r^{n-1}}{1-r} = \frac{1-\lim\limits_{n\to\infty} r^{n-1}}{1-r}$

Want to show: $\lim\limits_{n\to\infty} r^{n-1} = 0$

Correct me if I'm wrong, but

$\lim\limits_{n\to\infty} |r|^{n-1} = 0 \longrightarrow \lim\limits_{n\to\infty} r^{n-1} = 0$

Want to show: $\lim\limits_{n\to\infty} |r|^{n-1} = 0$

We know that $0 \leq |r| < 1$,

So,

for $1 <$ N $\in \mathbb{N}$, $|r|^N < |r|$ (assuming it is not the trivial case that r = 0)

Let: $\epsilon > 0$

$||r|^{n-1}$ - 0$| < \epsilon$

$|r|^{n-1}$ - 0 $< \epsilon$ (since that's always positive)

$|r|^{n-1} < \epsilon$

(n - 1) ln $|r| <$ ln ϵ

n ln $|r| < \epsilon +$ ln $|r$ $|$

n > $\frac{\epsilon + \ln |r|}{\ln |r|}$

So, if

n > $\frac{\epsilon + \ln |r|}{\ln |r|}$

Then,

$\forall\ \epsilon > 0, \exists$ N $\in \mathbb{N}$ st n $\geq$ N implies $||r|^{n-1} - 0| < \epsilon$

Hence,

$\lim\limits_{n\to\infty} r^{n-1} = 0$

Hence,

$\frac{1 - \lim\limits_{n\to\infty} r^{n-1}}{1-r} = \frac{1-0}{1-r} = \frac{1}{1-r}$

b. If we let the infinite repeating decimal 0.9999... stand for the limit:

$$\lim_{n\to\infty}\left(\frac{9}{10} + \frac{9}{10^2} + ... + \frac{9}{10^n}\right),$$

Show that 0.99999.... = 1.

From 10**(a)**,

$$\lim_{n\to\infty} 1 + r + r^2 + ...r^n = \frac{1}{1-r}$$

So,

$$\begin{aligned}\frac{9}{10} + \frac{9}{10^2} + ... + \frac{9}{10^n} &= 9\left(\frac{1}{10} + \frac{1}{10^2} + ... + \frac{1}{10^n}\right) \\ &= 9\left(\left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + ... + \left(\frac{1}{10}\right)^n\right)\end{aligned}$$

If we let r = $\frac{1}{10}$, then

$$\lim_{n\to\infty} 1 + \left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + ...\left(\frac{1}{10}\right)^n = \frac{1}{1-\frac{1}{10}}$$

So,

$$\begin{aligned}\lim_{n\to\infty} 9\left(\left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + ... + \left(\frac{1}{10}\right)^n\right) &= 9\lim_{n\to\infty}\left(\left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + ... + \left(\frac{1}{10}\right)^n\right) \\ &= 9\left(\frac{1}{1-\frac{1}{10}} - 1\right) \\ &= 10 - 9 \\ &= 1\end{aligned}$$

Hence,

0.9999999... = 1

Problem 13

Prove Lemma 4.3.11:
Every Cauchy sequence is bounded. (Similar to the proof of Theorem 4.1.13)

Proof.
Let: s_n be a Cauchy sequence
s_n is Cauchy if,
$\forall\ \epsilon > 0$, $\exists$ N $\in \mathbb{N}$ st for n, m $\geq$ N, $|s_n - s_m| < \epsilon$
With $\epsilon = 1$, we obtain N $\in \mathbb{N}$ st
$|s_n - s_m| < 1$ when n, m $\geq$ N
Thus, n $\geq$ N implies $|s_n| < |s_m| + 1$
If we let

$$M = max\{|s_1|, |s_2|, ...|s_N|, |s_m| + 1\}$$

Then we have $|s_n| \leq$ M $\forall$ n $\in \mathbb{N}$
Thus, (s_n) is bounded.

$\square$

Problem 14

Let (s_n) be the sequence defined by $s_n = (1 + \frac{1}{n})^n$.
Use the binomial theorem (Exercise 3.1.30) to show that (s_n) is an increasing sequence with $s_n < 3 \; \forall$ n.
Conclude that (s_n) is convergent. The limit of (s_n) is referred to as e and is used as the base for natural logarithms. The approximate value of e is 2.71828.

Let: $s_n = (1 + \frac{1}{n})^n$
Want to show: (s_n) is increasing, using the binomial theorem (Exercise 3.1.30)
$(1 + b)^n = 1 + nb + \frac{n(n-1)}{2!}b^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}b^r + ... + b^n$
So,
$(1 + (\frac{1}{n}))^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2!}(\frac{1}{n})^n + ... + \frac{n(n-1)...(n-(r-1))}{r!}(\frac{1}{n})^r + ... + (\frac{1}{n})^n$
In other words,

$$(1+\frac{1}{n})^n = \sum_{r=0}^{n} \binom{n}{r} (\frac{1}{n})^r$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \binom{n}{r} (\frac{1}{n})^r$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n!}{r!(n-r)!} (\frac{1}{n})^r$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n(n-1)(n-2)...(2)(1)}{(r)(r-1)...(2)(1)(n-r)(n-r-1)...(2)(1)} (\frac{1}{n})^r$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n(n-1)(n-2)...(2)(1)}{(r)(r-1)...(2)(1)(n-r)(n-r-1)...(2)(1)} \frac{1}{n^r}$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n(n-1)(n-2)...(n-(r-1))}{(r)(r-1)...(2)(1)} \frac{1}{n^r}$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n(n-1)...(n-(r-1))}{r!} \frac{1}{n^r}$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} \frac{n}{n}\frac{n-1}{n}...\frac{n-(r-1)}{n}\frac{1}{r!}$$

$$(1+\frac{1}{n})^n = 1 + \sum_{r=1}^{n} 1 * (1-\frac{1}{n}) * (1-\frac{2}{n})...(1-\frac{r-1}{n})\frac{1}{r!}$$

$$(1+\frac{1}{n+1})^{n+1} = 1 + \sum_{r=1}^{n+1} 1 * (1-\frac{1}{n+1}) * (1-\frac{2}{n+1})...(1-\frac{r-1}{n+1})\frac{1}{r!}$$

Let $i \in \{0, 1, 2, ... \; r-1\}$
We know that

$$n+1 \geq n$$

$$\frac{1}{n+1} \leq \frac{1}{n}$$

$$\frac{i}{n+1} \leq \frac{i}{n}$$

$$\forall i$$

So,

$$(1+\frac{1}{n})^n = 1+\sum_{r=1}^{n}\frac{1}{r!}\prod_i(1-\frac{i}{n})$$

$$(1+\frac{1}{n+1})^{n+1} = 1+\sum_{r=1}^{n}\frac{1}{r!}\prod_i(1-\frac{i}{n+1})+\frac{1}{r!}\prod_i(1-\frac{i}{n+1})$$

Since

$$\frac{i}{n+1}\leq\frac{i}{n}$$

$\forall$ i,

$$\sum_{r=1}^{n}\frac{1}{r!}\prod_i(1-\frac{i}{n})<\sum_{r=1}^{n}\frac{1}{r!}\prod_i(1-\frac{i}{n+1})+\frac{1}{r!}\prod_i(1-\frac{i}{n+1})$$

In other words:

$$(1+\frac{1}{n})^n<(1+\frac{1}{n+1})^{n+1}$$

$\forall$ n $\in \mathbb{N}$
Hence, s_n is increasing.
Want to show: (s_n) is bounded with $s_n < 3$ $\forall$ n, using the binomial theorem (Exercise 3.1.30)
Q(n): $s_n < 3$ $\forall$ n $\in \mathbb{N}$
$s_1 = 2 < 3$
$s_2 = 2.25 < 3$
Assume for k $\in \mathbb{N}$ that $s_k < 3$

$$s_k = (1+\frac{1}{k})^k < 3$$

Want to show: $s_{k+1} < 3$

$$(1+\frac{1}{k})^k < 3$$

$$1+\sum_{r=1}^{n}\frac{1}{r!}\prod_i(1-\frac{i}{n}) < 3$$

$$\forall n \in \mathbb{N}$$

$$\prod_i(1-\frac{i}{n}) < 1$$

(I think... I don't know where to go from here. How would you use $\frac{1}{1-r}$?)

Want to show: (s_n) is convergent
Since we just showed that s_n is increasing and bounded above (in theory), s_n is convergent by Theorem 4.3.3 (the Monotone Convergence Theorem)