

Homework Due 10/5/17 (7 problems):

Section 4.1 pages 169 - 170; 1, 6(b), 7(f), 9(a), 11, 12, 15

Test up to 4.2.4

(theorem 4.2.1 is a possibility on test)

#1

Mark each statement True or False. Justify each answer.

- a. If (s_n) is a sequence and $s_i = s_j$ then $i = j$.

False.

Let: $(s_n) = \{1^n\}$

- b. If $s_n \rightarrow s$, then, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$.

True.

A sequence $\{s_n\}$ is said to **converge** to $s \in \mathbb{R}$ provided that $\forall \epsilon > 0$

$\exists N \in \mathbb{N} \leq n$ st

$|s_n - s| < \epsilon$

This is the definition of convergence, so this implies that $s_n \rightarrow s$

- c. If $s_n \rightarrow k$ and $t_n \rightarrow k$, then $s_n = t_n \forall n \in \mathbb{N}$.

False.

Let: $k = 1$, $s_n = \frac{1}{n}$, $t_n = \frac{-1}{n}$

- d. Every convergent sequence is bounded.

True.

By Theorem 4.1.13, this is true.

6(b)

Definition 4.1.2

A sequence $\{s_n\}$ is said to **converge** to $s \in \mathbb{R}$ provided that $\forall \epsilon > 0$

$\exists N \in \mathbb{N}$ st

$n \geq N \implies |s_n - s| < \epsilon$

Using only definition 4.1.2, prove the following:

For $k > 0$, $k \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \left(\frac{1}{n^k}\right) = 0$

Proof.

Let: $\{s_n\} = \frac{1}{n^k}$

$$|s_n - s| = \left| \frac{1}{n^k} - 0 \right| = \left| \frac{1}{n^k} \right|$$

Want to show: $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ st $n \geq N \implies \left| \frac{1}{n^k} \right| < \epsilon$

Let: $\epsilon > 0$, $k \in \mathbb{R} > 0$

Want to show: $\exists N \in \mathbb{N}$ st $\left| \frac{1}{N^k} \right| < \epsilon$

Let: $\left| \frac{1}{N^k} \right| < \epsilon$

$$\frac{1}{|N^k|} < \epsilon$$

$$\frac{1}{\epsilon} < |N^k|$$

$$|N^k| = N^k \text{ since } N \in \mathbb{N} \text{ and } k > 0 \text{ (1)}$$

$$\frac{1}{\epsilon} < N^k$$

$$\left(\frac{1}{\epsilon}\right)^{\frac{1}{k}} < N$$

By the AP, there is an N bigger than $\left(\frac{1}{\epsilon}\right)^{\frac{1}{k}}$.

Hence, there is a natural number $N \forall \epsilon > 0$ st $n \geq N$ implies $\left| \frac{1}{n^k} \right| < \epsilon$

Thus, the limit of s_n is 0.

□

7(f)

Using any of the results in this section (4.1), prove the following:

If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$

Proof.

$|x| < 1$ implies $0 \leq |x| < 1$ **(1)**

Let: $s_n = x^n$, $s = 0$

Want to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|x|^N < \epsilon$

Let: $\epsilon > 0$

Want to show: $\exists N \in \mathbb{N}$ st $N \leq n$ implies $|x|^N < \epsilon$

$|x|^N < \epsilon$

$\ln |x|^N < \ln \epsilon$

$N \ln |x| < \ln \epsilon$

$N > \frac{\ln \epsilon}{\ln |x|}$

By the AP, there is an N greater than $\frac{\ln \epsilon}{\ln |x|}$

Hence, there is a natural number $N \forall \epsilon > 0$ st $n \geq N$ implies $|x|^N < \epsilon$

Thus, the limit of s_n is 0.

□

9(a)

For each of the following, prove or give a counter example:

If (s_n) converges to s , then $(|s_n|)$ converges to $|s|$.

Proof.

If s_n converges to s , then by definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $N \leq n$ implies $|s_n - s| < \epsilon$

Want to show: $||s_n| - |s|| < \epsilon$

Case 1: s_n and s are the same sign.

$||s_n| - |s|| = |s_n - s|$

Therefore, $N \leq n$ implies $||s_n| - |s|| < \epsilon$

If we let $s_n = |s_n|$ and $|s| = s$, then $|s_n|$ converges to $|s|$.

Case 2: s_n and s are different signs.

$||s_n| - |s|| \leq |s_n - s| < \epsilon$

$||s_n| - |s|| < \epsilon$

Therefore, $N \leq n$ implies $||s_n| - |s|| < \epsilon$

If we let $s_n = |s_n|$ and $|s| = s$, then $|s_n|$ converges to $|s|$.

Hence, result.

□

11

Given the sequence (s_n) , $k \in \mathbb{N}$, let (t_n) be the sequence defined by $t_n = s_{n+k}$. That is, the terms in (t_n) are the same as that of the terms in (s_n) after the first k terms have been skipped. Prove that (t_n) converges iff (s_n) converges, and if they converge, show that $\lim t_n = \lim s_n$. Thus, the convergence of a sequence is not affected by omitting (or changing) a finite number of terms.

Proof.

→

(t_n) converges → (s_n) converges

If t_n converges, then by definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|t_n - L| < \epsilon$

(or)

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $|t_n - L| < \epsilon \forall n \geq N$

Since $t_n = s_{n+k}$,

we know that s_{n+k} converges.

Let: $n_1 = n + k$

So,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n_1 \geq N$ implies $|s_{n_1} - L| < \epsilon$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n + k \geq N$ implies $|s_{n+k} - L| < \epsilon$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N - k$ implies $|s_{n+k} - L| < \epsilon$

Notice that $N - k \in \mathbb{N}$. Let's call it N_1

$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ st $n \geq N_1$ implies $|s_{n+k} - L| < \epsilon$

Since there is still a natural number N_1 st $n \geq N_1$ implies $|s_{n+k} - L| < \epsilon$,

If t_n converges, then s_n converges.

←

(s_n) converges → (t_n) converges

If s_n converges, then by definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - L| < \epsilon$

Since $t_n = s_{n+k}$, $t_{n-k} = s_n$

So since s_n converges, t_{n-k} converges.

If we let $n_1 = n - k$,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n_1 \geq N$ implies $|t_{n_1} - L| < \epsilon$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n - k \geq N$ implies $|t_{n-k} - L| < \epsilon$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq (N + k)$ implies $|t_{n-k} - L| < \epsilon$

Notice that $N + k \in \mathbb{N}$. Let's call it N_1

$\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ st $n \geq N_1$ implies $|t_{n-k} - L| < \epsilon$

Since there is still a natural number N_1 st $n \geq N_1$ implies $|t_{n-k} - L| < \epsilon$,

If s_n converges, then t_n converges.

□

12

a. Assume that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim(s_n t_n) = 0$.

If $\lim s_n = 0$, then by definition,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st if $n \geq N$, then $|s_n - 0| < \epsilon$

If t_n is a bounded sequence, then $\forall n \in \mathbb{N}$, $a \leq t_n \leq b$, where $a, b \in \mathbb{R}$

We know that t_n will always be between two constants a and b , so let's let t_n be some constant c dependent on n , where $a \leq c_n \leq b$.

Since s_n converges,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n - 0| < \epsilon$$

can be simplified to

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n| < \epsilon$$

Want to show: $\lim(s_n t_n) = 0$

$$\lim(s_n t_n) = 0 \text{ if}$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n t_n| < \epsilon$$

Since we let $t_n = c$, some bounded real number, this is equivalent to

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |c s_n| < \epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n| < |c| \epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n| < \epsilon_1$$

which is equivalent to

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n| < \epsilon$$

Hence, result.

- b. Show by example that the boundedness of (t_n) is a necessary condition in part (a).

If $\lim s_n = 0$, then by definition,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n - 0| < \epsilon$$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n| < \epsilon$$

However, if we let t_n be unbounded (i.e. let $t_n = e^n$), this doesn't work. See below:

$s_n t_n$ is bounded if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ st if } n \geq N, \text{ then } |s_n t_n| < \epsilon$$

Suppose: $s_n = \frac{1}{n}$

Then $s_n t_n = \frac{e^n}{n}$

Since e^n grows faster than $\frac{1}{n}$, $s_n t_n$ grows overall as n approaches infinity.

Hence, the boundedness of t_n is necessary.

15

- a. Prove that x is an accumulation point of a set S iff \exists a sequence (s_n) of points in $S \setminus \{x\}$ st (s_n) converges to x .

→

Let: $x \in S'$

This means that $N^*(x, \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$. **(1)**

$N^*(x, \epsilon)$ means $\{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}$

If $\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - x| < \epsilon$,

Then (s_n) converges to x .

Let: $s_n \in N^*(x, \frac{1}{n}) \cap S \neq \emptyset$

Then

$|s_n - x| < \frac{1}{n}$ and $s_n \in S \setminus \{x\}$

Let: $\epsilon > 0$

$\exists N(\epsilon) \in \mathbb{N}$ st $\frac{1}{N} < \epsilon$ (By AP)

Thus, from **(1)**,

$|s_n - x| < \epsilon, \forall n \geq N$.

Hence, $\lim s_n = x$ and $s_n \in S \setminus \{x\} \forall n \in \mathbb{N}$

←

Conversely,

Assume: $\{s_n\}$ is a sequence in $S \setminus \{x\}$ st $\lim_{n \rightarrow \infty} s_n = x$

Want to show: $x \in S'$

$\forall \epsilon > 0, \exists N(\epsilon)$ (as in N is chosen based on ϵ) $\in \mathbb{N}$ st

$|s_n - x| < \epsilon \forall n \geq N$ and $s_n \in S \setminus \{x\}$

$s_n \in (x - \epsilon, x + \epsilon), s_n \neq x$

Thus, $N^*(x, \epsilon) \cap S \neq \emptyset$

So, $x \in S'$

Hence result.

- b. Prove that a set S is closed iff, whenever (s_n) is a convergent sequence of points in S , it follows that $\lim s_n$ is in S .

→

Let: S be a closed set.

Want to show: (s_n) is a sequence in S st $\lim_{n \rightarrow \infty} s_n = s$ implies $s \in S$

If S is closed, then $S = \text{cl } S = S \cup S', \text{bd } S \subset S$

$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$ st $|s_n - s| < \epsilon \forall n \geq N$

Want to show: $s \in S$

Case

- i) $s \in S$.

In this case, we are done.

ii) $s \notin S$

Hence, (s_n) is a sequence in $S \setminus \{s\}$ st $\lim_{n \rightarrow \infty} s_n = s$. By **(a)**, $s \in S'$.

Since S is closed, $s \in S$.

Note: The above is what we did in class. Below (until \longleftarrow) is my original answer. Can you tell me if the next 11 or so lines are valid?

Suppose: $\lim s_n \notin S$

This implies that $s \notin S$ where

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $n \geq N$ implies $|s_n - s| < \epsilon$

Since S is closed,

Let: u be the closest boundary point to s

Now, let $\epsilon = |\frac{u-s}{2}|$

We know that $|s_n - s| < \epsilon$ for this epsilon.

$|s_n - s| < |\frac{u-L}{2}|$

Which implies that the distance between s_n and s is less than the distance between s_n and the nearest boundary point of S .

This means there is an s_n st $s_n \notin S$, a contradiction.

So, s_n is not a convergent sequence of points in S if $\lim s_n$ is not in S .

\longleftarrow

Conversely,

Assume: whenever (s_n) is a sequence in S st $\lim_{n \rightarrow \infty} s_n = s$, then $s \in S$

Want to show: S is closed

We will use Theorem 3.4.17 (a): S is closed iff $S' \subset S$

Let: $s \in S'$

$s \in S'$ means $\forall \epsilon > 0, N^*(s, \epsilon) \cap S \neq \emptyset$

Let: $s_n \in N^*(s, \frac{1}{n}) \cap S \neq \emptyset, n \in \mathbb{N}$

So,

$|s_n - s| < \frac{1}{n}, s_n \in S \forall n \in \mathbb{N}$

Hence $\lim_{n \rightarrow \infty} s_n = s$

We know $\exists N \in \mathbb{N}$ st $\frac{1}{N} < \epsilon$ by AP.

From **(1)**, $|s_n - s| < \frac{1}{n} < \epsilon \forall n \geq N$

Hence, $\lim_{n \rightarrow \infty} s_n = s, s_n \in S, \forall n \in \mathbb{N}$

By assuming $s \in S, S$ is closed.