

Assignment Set: 1, 2, 3, 4, 6, 8 from pages 148 - 149

1)

Mark each statement as true or false. Justify each answer.

- a. A set S is compact iff every open cover of S contains a finite subcover.

True.

The original definition is:

A set $S \subset \mathbb{R}$ is said to be compact if every open cover has a finite subcover.

\rightarrow (every open cover \Rightarrow compact)

Let: $S \subset \mathbb{R}$ be a set st every open cover has a finite subcover.

Then, by definition, S is compact.

\leftarrow (every open cover \Leftarrow compact)

Let: $S \subset \mathbb{R}$ be compact

Then, by definition, S is a set st every open cover has a finite subcover.

Hence, result.

- b. Every finite set is compact.

True.

Let: S be a finite set, $0 < |S| < \infty$, $S = \{s_1, s_2, \dots, s_n\}$

Let: $l = \min \{s_1, s_2, \dots, s_n\}$, $u = \max \{s_1, s_2, \dots, s_n\}$

Since S has lower bound l and upper bound u , S is bounded.

Since $l, u \in \text{bd } S$ by definition of maximum and minimum, S is closed.

Hence, by Heine-Borel, S is compact.

- c. No infinite set is compact.

Not true.

If $S = [0, 1]$, there are infinite values between 0 and 1.

However, $[0, 1]$ is compact.

- d. If a set is compact, then it has a maximum and a minimum.

True.

Let: $S \neq \emptyset$ be a compact set

Since S is compact, S is also closed and bounded.

Since $S \neq \emptyset$, there is at least one element.

Since S is closed, $\text{bd } S \subset S$.

Since there is at least one element, and all boundary points of S are in S , there is a maximum and minimum element.

e. If a set has a maximum and a minimum, then it is compact.

Let: $S \neq \emptyset$, $l = \min S$, $u = \max S$

Since $l, u \in \text{bd } S$ by definition of minimum and maximum, S is bounded.

Notice that $l, u \notin \text{int } S$.

Since $\exists s \in S$ st $s \notin \text{int } S$, S is not open.

Therefore, S is closed.

By the Heine-Borel theorem, since S is closed and bounded, S is compact.

2)

Mark each statement as true or false. Justify each answer.

a. Some unbounded sets are compact.

False.

By Heine-Borel, S is compact only if closed and bounded.

b. If $S \subset \mathbb{R}$ is compact, then $\exists x \in \mathbb{R}$ st $s \in S'$

False.

The empty set is compact and contains no elements.

c. If S is compact and $s \in S'$, then $s \in S$.

True.

By Heine-Borel, if S is compact, then S is closed and bounded.

If S is closed, then $S' \subset S$ by Theorem 3.4.17

So, if $s \in S'$, then $s \in S$

d. If S is unbounded, then S has at least one accumulation point.

False.

\mathbb{N} is a counter example.

e. **Let:** $F = \{A_i, i \in \mathbb{N}\}$. Suppose that the intersection of any finite subfamily of F is nonempty. If $\bigcap F = \emptyset$, then, for some $k \in \mathbb{N}$, A_k is not compact.

True.

Suppose: $\forall A_i \in F$, A_i is compact

Then, by Heine-Borel, A_i is closed and bounded.

If any finite subfamily of F is nonempty, but $\bigcap F = \emptyset$, then $\lim_{i \rightarrow \infty} F = \emptyset$, so F gets smaller as i grows.

However, as F gets smaller, since A_i is closed, $\text{bd } A_i \subset A_i$, which means eventually F should converge to a set with a single element in it.

But this is not true, so $\exists A_i \in F$ st A_i is not compact.

3)

Show that each subset of \mathbb{R} is not compact by describing an open cover for it that has no finite subcover.

a. $[1, 3)$

$$\{n \in \mathbb{N} : \bigcup_{i=1}^n (0, 2 + \sum_{k=1}^i \frac{1}{2^k})\}$$

$$\text{Simpler solution: } \bigcup_{n=1}^{\infty} (0, 3 - \frac{1}{n})$$

b. $[1, 2)$

$$\{n \in \mathbb{N} : \bigcup_{i=1}^n ((2 - \sum_{i=1}^n \frac{1}{2^i}), (3 + \sum_{i=1}^n \frac{1}{2^i}))\}$$

$$\text{Simpler solution: } \bigcup_{n=1}^{\infty} (0, 2 - \frac{1}{n})$$

c. \mathbb{N}

$$\{\bigcup_{n=1}^{\infty} (0, n)\}$$

$$\text{Other solution: } \bigcup_{n=1}^{\infty} (n - \frac{1}{2}, n + \frac{1}{2})$$

d. $\{\frac{1}{n} : n \in \mathbb{N}\}$ (Closed in \mathbb{R} , but not closed in \mathbb{Q})

$$\{n \in \mathbb{N} : \bigcup_{i=1}^n (0, \sum_{k=1}^i \frac{1}{2^k})\}$$

$$\text{Simpler solution: } \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n})$$

e. $\{x \in \mathbb{Q} : 0 \leq x \leq 2\}$

but wait, isn't this a closed and bounded set? It's obviously bounded, so why is that not closed?

Because the set only contains rational numbers and there are some irrational numbers that are accumulation points. A set has to contain all of its accumulation points to be closed.

In any case, here's a cover:

$$\bigcup_{i=0}^{\infty} (-1, \sum_{k=0}^i \binom{2k}{k} \frac{1}{8^k}) \cup (\sqrt{2} - (\sqrt{2} - \sum_{k=0}^i \binom{2k}{k} \frac{1}{8^k}), 3)$$

$$\text{Other solution: } \bigcup_{n=1}^{\infty} [(-1, \sqrt{2}), \cup (\sqrt{2} + \frac{1}{n}, 3)]$$

4)

Prove that the intersection of any collection of compact sets is compact.

Let: S be $\bigcap_{\alpha \in I} G_{\alpha}$ where I is an index set and all G_{α} 's are compact

Let: S be nonempty (since if it's empty, then it's compact anyhow)

By Heine-Borel, G_{α} is both closed and bounded $\forall \alpha$.

Let: U be the set of all least upper bounds from each G_{α} , and L be the set of all greatest lower bounds from each G_{α} 's

Since G_{α} is closed $\forall G_{\alpha}$, each element in U is a max, and each element in L is a min.

Since S is an intersection, its minimum will be max L , and its maximum will be min U , which we know exists because $S \neq \emptyset$

Since S has a min and a max, the min and max are members of its boundary set.

Since the min and max are members of the boundary set, they can't be interior points.

Since $\exists s \in S$ that isn't an interior point, S is not open and therefore closed.

Since S has a minimum and a maximum, it is also bounded.

Since S is closed and bounded, it is compact by Heine-Borel.

6)

Show that compactness is necessary in Corollary 3.5.8. That is, find a family of intervals $\{A_n : n \in \mathbb{N}\}$ with $A_{n+1} \subset A_n \forall n$, $\bigcup_{n=1}^{\infty} A_n = \emptyset$, and such that:

- a. The sets A_n are all closed.

$$\{n \in \mathbb{N} : \emptyset\}$$

- b. The sets A_n are all bounded.

$$\{n \in \mathbb{N} : (0, 0)\}$$

8)

If $S \subset \mathbb{R}$ is compact and $T \subset S$ is closed, then T is compact.

- a. Prove this using the definition of compactness.

If S is compact, then $\forall \bigcup_{\alpha \in I} G_\alpha$ that cover S (G_α is open), $\exists \bigcup_{i=1}^n G_{\alpha_i}$ - a finite subcover of S .

Let: $\mathcal{C}\mathcal{V}$ be an arbitrary open cover for S , $\mathcal{S}\mathcal{C}\mathcal{V}$ be $\mathcal{C}\mathcal{V}$'s finite subcover for S

By definition, $S \subset \mathcal{C}\mathcal{V}$ and $S \subset \mathcal{S}\mathcal{C}\mathcal{V}$

Since $T \subset S$, $T \subset \mathcal{C}\mathcal{V}$ and $T \subset \mathcal{S}\mathcal{C}\mathcal{V}$

So, $\forall \mathcal{C}\mathcal{V}$ of S and $\forall \mathcal{S}\mathcal{C}\mathcal{V}$ of S , $\mathcal{C}\mathcal{V}$ and $\mathcal{S}\mathcal{C}\mathcal{V}$ are covers and finite subcovers for T .

So, T is compact.

- b. Prove using the Heine-Borel theorem: If $S \subset \mathbb{R}$ is compact and $T \subset S$ is closed, then T is compact.

Since S is compact, by Heine-Borel, S is closed and bounded.

Let: $T \neq \emptyset$ (since if $T = \emptyset$ then it is compact anyhow)

Since T is closed, $\text{bd } T \subset T$.

Since $T \neq \emptyset$ and $\text{bd } T \subset T$, T contains a maximum and a minimum.

Therefore, T is bounded.

Since T is both closed and bounded, T is compact.