

Assignment Set: 6, 7, 15, 17, 19, 21 from pages 141 - 142

6)

Find the closure of each set:

a. $\{ \frac{1}{n} : n \in \mathbb{N} \}$

Answer: \emptyset

b. \mathbb{N}

Answer: \mathbb{N}

c. \mathbb{Q}

Answer: \mathbb{R}

d. $\bigcap_{n=1}^{\infty} (0, \frac{1}{n})$

Answer: \emptyset

e. $\{ x : |x - 5| \leq \frac{1}{2} \}$

$[4.5, 5.5]$

Answer: $[4.5, 5.5]$

f. $\{ x : x^2 > 0 \}$

$(0, \infty)$

Answer: $[0, \infty)$

7)

Let $S, T \subset \mathbb{R}$. Find a counterexample of each of the following:

a. If P is the set of all isolated points of S , then P is a closed set.

Answer: Let $S = \mathbb{N}$

b. Every open set contains at least two points.

Answer: \emptyset

c. If S is closed, then $\text{cl}(\text{int } S) = S$.

Answer: Let $S = \mathbb{Q}$

d. If S is open, then $\text{int}(\text{cl } S) = S$.

Answer: Let $S = (-1, 0) \cup (0, 1)$

e. $\text{bd}(\text{cl } S) = \text{bd } S$

Answer: Let $S = (-1, 0) \cup (0, 1)$

f. $\text{bd}(\text{bd } S) = \text{bd } S$

Answer: Let $S = \mathbb{Q}$. Then $\text{bd } S$ is \mathbb{R} , and $\text{bd}(\text{bd } S) = \emptyset \neq \mathbb{R}$.

g. $\text{bd}(S \cup T) = (\text{bd } S) \cup (\text{bd } T)$

Answer: Let $S = \mathbb{R}$, $T = (0, 1)$. $\text{bd}(S \cup T) = \emptyset$, but $\text{bd } S \cup \text{bd } T = \emptyset \cup \{0, 1\}$

h. $\text{bd}(S \cap T) = (\text{bd } S) \cap (\text{bd } T)$

Answer: Let $S = (0, 1)$, $T = (1, 2)$. $\text{bd}(S \cap T) = \emptyset$, but $\text{bd } S \cap \text{bd } T = 1$.

15)

Prove: If x is an accumulation point of the set S , then every neighborhood of x contains infinitely many points of S .

Proof.

Suppose that \exists a deleted neighborhood of x , called N , that contains n points x_1, x_2, \dots, x_n of S where n is a finite amount and $x_1 \leq x_2 \leq \dots \leq x_n$

x is an accumulation point on S if $\forall \epsilon > 0, N^*(x, \epsilon) \cap S \neq \emptyset$.

N is a deleted neighborhood of S if $\forall x \in \{y \in \mathbb{R} : 0 < |y - x| < \epsilon\}, x \in N$.

Let $\hat{\epsilon} = \epsilon + \epsilon$, and $x_0 = x_1 - \hat{\epsilon}$.

By definition, $x_0 \in N$, since N is a neighborhood $\forall \epsilon > 0$.

However, N only has n elements. A contradiction.

So, N can't be a deleted neighborhood since it has a finite number of elements, which means x can't be an accumulation point. □

17)

Prove: S' is a closed set.

Proof.

By definition, $\forall x \in S', \epsilon > 0, N^*(x, \epsilon) \cap S \neq \emptyset$

Notice that if S' is empty or S' is \mathbb{R} , then S' is a closed set and we are done.

Let: $\mathbb{R} \setminus S' \subset \mathbb{R}, x \in \mathbb{R} \setminus S'$

Want to show: $\mathbb{R} \setminus S'$ is open.

$\mathbb{R} \setminus S'$ is open iff $\mathbb{R} \setminus S' = \text{int}(\mathbb{R} \setminus S')$

$\text{int} \mathbb{R} \setminus S' = \{s: N(s, \epsilon) \subset \mathbb{R} \setminus S'\}$ □

19)

Suppose S is a nonempty bounded set and let $m = \sup S$. Prove or give a counter example: m is a boundary point of S .

Proof.

By definition,

$s \leq m, \forall s \in S$, and,

$\forall \epsilon > 0, \exists s' \in S$ st $m - \epsilon < s'$

By the second part of the definition of the supremum of S , $N(m, \epsilon) \cap S \neq \emptyset$.

Notice also that, by the first part of the definition of the supremum of S , $(m + \epsilon) \notin S$. This means that $N(m, \epsilon) \cap \mathbb{R} \setminus S \neq \emptyset$.

By definition, m is a boundary point. □

21)

Let A be a nonempty open subset of \mathbb{R} and let $Q \subset \mathbb{Q}$. Prove: $A \cap Q \neq \emptyset$.

Proof.

Notice that $Q \subset \mathbb{Q} \subset \mathbb{R}$.

Since A is nonempty, \exists at least one element $a \in \mathbb{R}$.

Since A is nonempty and open, $a + \epsilon \in A$.

If $a \in Q$, then result.

If $a + \epsilon \in Q$, then result.

If $a \notin Q$ and $(a + \epsilon) \notin Q$, then:

Let $x = a$, $y = a + \epsilon$, $z = y - x$.

By Archimedes' axiom, $\exists n \text{ st } n > \frac{1}{z}$

$nz > 1$

$ny - nx > 1$

Since the difference between ny and nx is bigger than 1,

$\exists m \in \mathbb{Z}$ st $nx < m < ny$.

See that since $x < \frac{m}{n} < y$, $\frac{m}{n}$ is a rational number, and $\frac{m}{n} \in A$.

Hence, result. □