HW 11: page 220 - 221, #1, 2, 5 and page 226-227, # 1 - 3, 4(a)(b), 5, 11

Exercise 1 (pages 220 - 221)

Mark each statement True or False. Justify each answer.

a. Let D be a compact subset of $\mathbb{R}$ and suppose that f : D $\longrightarrow \mathbb{R}$ is continuous. Then f(D) is compact.

   True, by Theorem 5.3.2.

b. Suppose that f : D $\longrightarrow$ R is continuous. Then, there exists a point x_1 in D st $f(x_1) \geq f(x)\ \forall\ x \in D$

   False.

   Let: $f(x) = x$ and $D = \mathbb{R}$

   Suppose: $\exists\ x_1 \in D$ st $f(x_1) \geq f(x)\ \forall\ x \in D$

   Notice that $(f(x_1) + 1) \in \mathbb{R}$, and if $x_2 = (f(x_1) + 1)$, then $f(x_2) = (f(x_1) + 1) > f(x_1)$. A contradiction.

c. Let D be a bounded subset of $\mathbb{R}$ and assume that f : D $\longrightarrow \mathbb{R}$ is continuous. Then f(D) is bounded.

   False.

   Let: $f : (0, \infty) \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$

   Suppose: $\exists\ f(x_1)$ st $f(x_1) \geq f(x)\ \forall\ x \in (0, \infty)$

   Notice that $(f(x_1) + 1) \in \mathbb{R}$, and if $x_2 = \frac{1}{f(x_1)+1}$, then $f(x_2) = (f(x_1) + 1) > f(x_1)$. A contradiction.

Exercise 2 (pages 220 - 221)

Mark each statement True or False. Justify each answer.

a. Let f : [a,b] $\longrightarrow \mathbb{R}$ be continuous and assume $f(a) < 0 < f(b)$. Then there exists a point $c \in (a, b)$ st $f(c) = 0$.

   True, by Theorem 5.3.6 (IVT).

b. Let f : [a,b] $\longrightarrow \mathbb{R}$ be continuous and assume $f(a) \leq k \leq f(b)$. Then there exists a point $c \in [a,b]$ st $f(c) = k$.

   True, by Theorem 5.3.6 (IVT). Also because this statement is just (a) above with k = 0, except weaker.

c. If f : D $\longrightarrow \mathbb{R}$ is continuous and bounded on D, then f assumes maximum and minimum values on D.

   False.

   Let: $f : (0, 1) \longrightarrow \mathbb{R}$ be defined by $f(x) = x$

   Suppose: f has $x \in D$, a maximum value on D

   Notice that $0 < x < 1$, and that $x < x + \frac{1-x}{2}$.

   However, notice also that $x + \frac{1-x}{2} < 1$

   But x is a maximum value on D. A contradiction.

   WLOG, a minimum value on D is similar.

Exercise 5 (pages 220 - 221)

Show that the equation $5^x = x^4$ has at least one real solution.
Let: $f : [-1, 0] \longrightarrow \mathbb{R}$ be defined by $f(x) = 5^x - x^4$
Notice that $f(-1) = -0.8$ and $f(0) = 1$
Since $5^x - x^4 = 0$ means $5^x = x^4$, and $-0.8 < 0 < 1$,
by Theorem 5.3.6, since f(x) is continuous on $\mathbb{R}$,
$\exists$ c $\in [-1, 0]$ st f(c) = 0.

Exercise 1 (pages 226 - 227)

Let $f : D \longrightarrow \mathbb{R}$. Mark each statement True or False. Justify each answer.

a. f is uniformly continuous on D iff for every $\epsilon > 0$ there exists a $\delta > 0$ st $|f(x) - f(y)| < \delta$ whenever $|x - y| < \epsilon$ and x, y $\in$ D.

   This isn't the definition, but I can't find a counter example for it...

b. If D = {x}, then f is uniformly continuous at x.

   True. Since x is the only element in the domain, and since f is a function, f(x) is the only element in the range of f which makes $|f(x) - f(y)|$ always less than any $\epsilon > 0$ since there is only one object in the range, making them the same object in any possible case.

c. If f is continuous and D is compact, then f is uniformly continuous on D.

   True, by Theorem 5.4.6.

Exercise 2 (pages 226 - 227)

Let $f : D \longrightarrow \mathbb{R}$. Mark each statement True or False. Justify each answer.

a. In the definition of uniform continuity, the positive δ depends only on the function f and the given $\epsilon > 0$.

   False. The positive δ depends on the given x, y $\in$ D as well.

b. If f is continuous and (x_n) is a Cauchy sequence in D, then $(f(x_n))$ is a Cauchy sequence.

   False.

   Let: $x_n = \frac{1}{n}$, $n \in \mathbb{N}$ and $f : (0, 1] \longrightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$

   Notice that $f(x_n) = 1, 2, 3...$

   This is not a Cauchy sequence.

c. If $f : (a, b) \longrightarrow \mathbb{R}$ can be extended to a function that is continuous on [a, b], then f is uniformly continuous on (a,b).

   True, by Theorem 5.4.9.

Exercise 3 (pages 226 - 227)

Determine which of the following continuous functions are uniformly continuous on the given set. Justify your answers.

a. f(x) = x on [2, 5] **since f is continuous and D is compact, f is uniformly continuous (by Theorem 5.4.6)**

b. f(x) = x on (0, 2) **since $\widetilde{f}$: [0, 2] $\longrightarrow \mathbb{R}$ is continuous, f is uniformly continuous (by Theorem 5.4.9)**

c. f(x) = x^2 + 2x − 7 on [0, 5] **since f is continuous and D is compact, f is uniformly continuous (by Theorem 5.4.6)**

d. f(x) = x^2 + 2x − 7 on (1, 4) **since $\widetilde{f}$: [1, 4] $\longrightarrow \mathbb{R}$ is continuous, f is uniformly continuous (by Theorem 5.4.9)**

e. f(x) = $\frac{1}{x^2}$ on (0, 1) **Since $\lim_{x\to 0}$ f(x) does not exist, f(x) cannot be extended to a continuous function. Therefore, f is not uniformly continuous.**

f. f(x) = $\frac{1}{x^2}$ on (0, ∞) **Since $\lim_{x\to 0}$ f(x) does not exist, f(x) cannot be extended to a continuous function. Therefore, f is not uniformly continuous.**

g. f(x) = $\frac{x^2-4}{x-2}$ on (2, 4) **Since $\lim_{x\to 2}$ f(x) and $\lim_{x\to 4}$ f(x) exist, f(x) can be extended to a continuous function. Therefore, f is uniformly continuous.**

h. f(x) = x sin($\frac{1}{x}$) on (0, 1) **Since $\lim_{x\to 0}$ f(x) = 0 and $\lim_{x\to 1}$ f(x) = sin(1), f(x) can be extended to a continuous function. Therefore, f is uniformly continuous.**

Exercise 4(a)(b) (pages 226 - 227)

Prove that each function is uniformly continuous on the given set by directly verifying the ϵ - δ property in Definition 4.1.

Definition 5.4.1:

f : D $\longrightarrow \mathbb{R}$ is uniformly continuous on D if

$\forall\ \epsilon > 0, \exists\ \delta > 0$ st $0 < |x - y| < \delta$ and x, y $\in$ D implies $|f(x) - f(y)| < \epsilon$

a. f(x) = x^3 on [0, 2]

$\forall\ \epsilon > 0, \exists\ \delta > 0$ st $0 < |x - y| < \delta$ and x, y $\in$ D implies $|x^3 - y^3| < \epsilon$

$$|x^3 - y^3|$$

$$|(x-y)(x^2+xy+y^2)|$$

$$|(x-y)(x^2+xy+y^2)| \leq |(x-y)|(|x^2|+|xy|+|y^2|) \leq 12|(x-y)| < \epsilon$$

so, whenever $|x - y| < \delta = \frac{\epsilon}{12}$, $|x^3 - y^3| < \epsilon$

b. f(x) = $\frac{1}{x}$ on $[2, \infty)$

$\forall\ \epsilon > 0, \exists\ \delta > 0$ st $0 < |x - y| < \delta$ and x, y $\in$ D implies $|\frac{1}{x} - \frac{1}{y}| < \epsilon$

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{y-x}{xy}|$$

Since all elements in the domain are positive,

$$|\frac{y-x}{xy}| = |y-x|\frac{1}{xy} = |x-y|\frac{1}{xy} < \epsilon$$

So, since $\frac{1}{x}$ is maximum at x = 2 and $\frac{1}{y}$ is maximum at y = 2,

$$|x-y| < xy\epsilon$$

$$|x-y| < (2)(2)\epsilon$$

$$|x-y| < \delta = 4\epsilon$$

so, whenever $|x - y| < \delta = 4\epsilon$, $|\frac{1}{x} - \frac{1}{y}| < \epsilon$

Exercise 5 (pages 226 - 227)

Prove that f(x) = $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Let: $\epsilon > 0$

Choose δ = SOMETHING to make $|x - y| < \delta$. We know that:

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|} < \frac{\delta}{|\sqrt{x}+\sqrt{y}|}$$

So, let

$$\delta\frac{1}{|\sqrt{x}+\sqrt{y}|} = \epsilon \Rightarrow \delta\frac{1}{|1+1|} = \epsilon \Rightarrow \delta = 2\epsilon$$

then $|\sqrt{x} - \sqrt{y}| < \epsilon$ if $|x - y| < \delta$ and x, y $\in [0, \infty)$

Exercise 11 (pages 226 - 227)

Let f : D $\longrightarrow \mathbb{R}$ be uniformly continuous on the bounded set D. Prove that f is bounded on D.
Use Theorem 5.4.8. The hint is that it's bounded.

Theorem 5.4.8

Let: f : D $\longrightarrow \mathbb{R}$ be uniformly continuous on D
Assume: $\{x_n\}$ is a Cauchy sequence in D
Then,
$\{f(x_n)\}$ is a Cauchy sequence.

Lemma 4.3.11

Every Cauchy sequence is bounded.

Proof strategy:

Any Cauchy sequence x_n in D means that $\{f(x_n)\}$ is a Cauchy sequence, and if $\{f(x_n)\}$ is a Cauchy sequence then it's bounded.
So, our strategy will be to somehow make a Cauchy sequence x_n that has a limit at c such that f(c) = max(f(D)) and, WLOG, d such that f(d) = min(f(D)).
Either that, or figure out a way to make a list of Cauchy sequences that hit all values in the domain.
Or maybe just prove it by contradiction:

Proof.

Suppose: f is NOT bounded on D
Then $\exists$ m $\in \mathbb{R}$ st f(x) > m $\forall$ x $\in$ D (or, WLOG, st f(x) < m $\forall$ x $\in$ D)
Since sup f(D) is unbounded above, there exists a monotone subsequence s_n in f(D) st $\lim_{n\to\infty} s_n = \infty$
This also means every subsequence of s_n diverges to infinity.
If we let t_n be a Cauchy sequence in D, then by Theorem 5.4.8, $\{f(t_n)\}$ must be a Cauchy sequence.
Recall the definition of uniform continuity:
$\forall\ \epsilon > 0, \exists\ \delta > 0$ st $|x - y| < \delta$ and x, y $\in$ D implies $|f(x) - f(y)| < \epsilon$
I know it's a jumble of statements... I think I need to stick those together somehow but I'm lost.

□

Exercise 11 (pages 226 - 227 (TEACHER VERSION))

Let f : D $\longrightarrow \mathbb{R}$ be uniformly continuous on the bounded set D. Prove that f is bounded on D (i.e. that |f(x)| < M, $\forall$ x $\in$ D)

Proof.

Suppose: f(D) is unbounded.
Then, for each n $\in \mathbb{N}$, $\exists$ x_n $\in$ D st

$$|f(x_n)| > n$$

Since $\{x_n\}$ is a bounded sequence, by the Bolzano-Weierstrauss Theorem, $\{x_{n_k}\}$ converges.
So, since $\{x_{n_k}\}$ converges, $\{x_{n_k}\}$ is Cauchy.
Since $\{x_{n_k}\}$ is Cauchy, $\{f(x_{n_k})\}$ is Cauchy, which is bounded. But f(D) is unbounded, a contradiction.
Hence, $\exists$ m $\in \mathbb{N}$ st

$$|f(x_{n_k})| \leq m$$

$\forall$ k $\in \mathbb{N}$ □